GLOBAL INVERSE THEOREMS OF GENERAL ORDER IN SIMULTANEOUS APPROXIMATION BY LINEAR COMBINATIONS OF CERTAIN LINEAR POSITIVE OPERATORS

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by

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TO MY

MOTHER

CERTIFICATE

It is certified that the work contained in the thesis entitled " GLOBAL INVERSE THEOREMS OF GENERAL ORDER IN SIMULTANEOUS APPROXIMATION BY LINEAR COMBINATIONS OF CERTAIN LINEAR POSITIVE OPERATORS" by "KUMAR SHAILESH SINGH'' has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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CONTENTS

CHAPTER I

INTRODUCTION

The present work is devoted to the study of simultaneous approximation properties of linear combinations of certain linear positive operators. The operators that we deal with in the thesis are Szász-Mirakjan-Hille operators, Bernstein polynomials and Bernstein-Kantorovitch polynomials.

For a sequence $\{L_n\}$, $n \in \mathbb{N}$ the set of natural numbers, of linear positive operators defined on a domain D of functions, we say that $\{L_n\}$ has simultaneous approximation property of order $k \in \mathbb{N}$, if for an arbitrary $f \in D$, the k^{th} derivative of $L_n f(x)$ converges to that of $f(x)$ provided the latter exists.

Following the work of Lorentz[50], on simultaneous approximation by Bernstein polynomials, many workers have studied simultaneous approximation property for various other linear positive operators. A survey of the representative work in this direction will be given later. Rathore and Agarwal[67] obtained local inverse and saturation theorems for simultaneous approximation by linear combinations of exponential type operators. But the corresponding global results have not been established as yet. Here we obtain global inverse theorems for simultaneous approximation by linear combinations of Szász

operators, Bernstein polynomials and Bernstein-Kantorovitch polynomials.

For the convolution type operators, such as those of Shapiro[73], Sikkema and Rathore[77], Kunwar[45], Rathore and Singh[68] etc., the same analysis as done for ordinary approximation carries over for simultaneous approximation, but such is not the case for the summation type operators that have been taken up for the study here. The reason for this lies in the fact that the operation of taking derivatives of the convolution type operators can be transferred to the function inside the integral sign, without affecting the convolution structure of the operator while this is not true for the operators being considered here. Thus, necessitating a closer observation and a more detailed analysis for the operators of the type understudy.

Inverse theorems of a general order in simultaneous approximation by linear combinations of the above mentioned operators are obtained, following the techniques of Berens and Lorentz[15], Ditzian[25] and Wimslin[92]. The main constituents being the Bernstein type inequalities for the respective operators, direct theorems, on appropriate spaces for simultaneous approximation by linear combinations of the operators and a generalization, due to Avadhani[7], of the Berens and Lorentz[15] lemma.

1.1 GLOBAL APPROXIMATION

An approximation theorem, valid for the whole domain of definition of the functions under consideration, is called a global approximation theorem.

It was not until Timan[85] observed that, if in the place of uniform approximation by algebraic polynomials we consider the point wise approximation on the given segment then conclusions could be reached on the whole domain rather than on contracting intervals and the remarkable work of Lorentz[51] on the global saturation theorem of Bernstein polynomials, that the work in this direction picked up the momentum. The results of the above authors aroused the interest of many more mathematicians and global approximation results for various type of operators have been obtained since then. Some of the representative works in this direction are those of Becker[8], Becker, Lautner, Nessel and Worms[11], Becker and Nessel[12,13,14], Berens and Lorentz[15], Ditzian[25], Meir[53], Reimenschneider[69], Sato[72], Totik[89,90], Winslin[92], Zhou[99] and others.

1.2 LINEAR COMBINATION

The operator sequences like Bernstein polynomials, Szász-Mirakjan-Hille operators, Baskakov operators, Meyer-König-Zeller operators and Bernstein Kantorovitch polynomials are conceptually easier to construct, but they lack in the rapidity of convergence for very smooth functions. In this regard we have the well known theorem of Korovkin[44] which

states that the optimal rate of convergence for any sequence of linear positive operators is at most $O(n^{-2})$.

In view of this it was pertinent to evolve some way out so that better order of approximation for smoother functions could be obtained for such operators. One such way is that of taking linear combinations of the above mentioned operators. Butzer[20] was one of the first to consider such combinations and he came up with results that showed that higher order of approximation could be achieved for smoother functions provided proper linear combinations were considered. Leviatan and Müller[48] also studied approximation by linear combinations of Gamma operators, but they considered linear combinations of very low order. It was only after the works of Rathore[65] and May[55] that more researchers started working in this direction. May[55] obtained local inverse and saturation theorems for the linear combinations of exponential type operators, Rathore and Agarwal[67] established these results for simultaneous approximation as well. Ditzian[25] proved a global inverse theorem for linear combinations of Bernstein polynomials, similar results were later obtained by Winslin[92] for the linear combinations of Bernstein-Kantorovitch polynomials.

Some further studies on linear combinations of various linear positive operators include Agarwal[1], Agarwal and Kasana[3], Agarwal,Sinha and Kasana[5], Avadhani[7], Ditzian and Ivanov[30], Sinha[83], Voght[91], Wood[93,94,95] and others.

1.3 ORDER OF APPROXIMATION

Most of the results, particularly on inverse theorems for linear combinations of linear positive operators $\{L_n\}$, available in the literature, concern with the order $O(n^{-\alpha})$ of approximation, α being a real number(see May[55], Ditzian[25], Agarwal[1], Rathore and Agarwal[67], Sinha[83], Rathore and Singh[68], Winslin[92] etc.).

Avadhani[7] gave a class Φ_r of functions and obtained order of approximation results with the order $O(\phi(n^{-1/2}))$ where $\phi(t)$ belongs to the above mentioned class. The class Φ_r is defined as follows:

Definition 1.3.1[7]: For any $r \in \mathbb{N}$, a positive function ϕ on $(0,c)$ is said to belong to the class Φ_r if it satisfies

(i) for any $0 < h < 1$, there exists a constant $K_\phi(h)$ depending only on ϕ and h such that for all $t \in (0,c]$

$$\frac{\phi(t)}{\phi(ht)} \leq K_\phi(h)$$

(ii) for any $\delta > 0$

$$K_\delta = \sup_{h \geq \delta} K_\phi(h) < \infty$$

(iii) $h^r K_\phi(h)$ tends to zero as h goes to zero .

Observe that $\phi(t) = t^\alpha$ (which corresponds to the approximation order of the type $n^{-\alpha}$) satisfies the above conditions for any integer $r > \alpha$. Other examples of $\phi(t)$ are

i) $t^\alpha\left[\log^p(\frac{1}{t})\right]^q$, $\alpha < r$

ii) $t^\alpha\left[\log^p(\frac{1}{t})\right]^{-q}$, $\alpha < r$.

Inverse theorems with respect to a general order of this type have been obtained in the present work.

1.4 SIMULTANEOUS APPROXIMATION

The present section briefly summarizes some of the work on simultaneous approximation.

Lorentz[50] was one of the first to consider this type of approximation, when he showed that:

"If $f(x)$ is bounded in $[0,1]$ and $f^{(k)}(x_o)$ exists at a point $0 \le x_o \le 1$, then $B_n^{(k)}f(x_o)$ converges to $f^{(k)}(x_o)$."

This was followed by the works of Minkova[59,60] and Sendov and Popov[74,75] on linear operators, Lupas and Müller[52] and Rathore[66] on M_n operators of Meyer-Konig-Zeller, Martini[54] on Baskakov operators and others.

Hasson[37] and Leviatan[47] obtained some results in simultaneous approximation, about the polynomials of best approximation. Leviatan[47] proved the following theorem:

"Let $f \in C^r[-1,1]$ and P_n be its n^{th} polynomial of best approximation on $[-1,1]$. Then for each $0 \le k \le r$

i) $\|f^{(k)} - P_n^{(k)}\| \le C_r\, n^k\, E_{n-k}(f^{(k)})$, $\quad n \ge k$

$\le K_r\, n^{2k-r}\, E_{n-r}(f^{(r)})$, $\quad n \ge r$

ii) if $-1 < \alpha < \beta < 1$ and $\|f-p\|_{[\alpha,\beta]} = \sup\limits_{\alpha \le x \le \beta} |f(x)-p(x)|$

$$= \sum_{i=0}^{r-1} C(i,r) \frac{(n_i+1)!}{(n_i-m)!} \frac{1}{m!} \sum_{s=0}^{m+1} \binom{m+1}{s} (-1)^{m+1-s} \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x)$$

$$\int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} \left[\frac{k+s}{n_i+1} - t\right]^m (f^{(m)}(t) - f^{(m)}(x))dt$$

$$+ \sum_{i=0}^{r-1} C(i,r)\left[\frac{(n_i+1)!}{(n_i-m)!} \frac{1}{m!} \sum_{s=0}^{m+1} \binom{m+1}{s} (-1)^{m+1-s} \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x)\right.$$

$$\left. \cdot \int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} \left(\frac{k+s}{n_i+1} - t\right)^m dt - 1 \right] f^{(m)}(x)$$

(5.4.1) $=S_1(x) + S_2(x)$, say.

By lemma 3.2.1 and (3.2.1),

$$S_2(x) = \sum_{i=0}^{r-1} C(i,r)\left[\frac{(n_i+1)!}{(n_i-m)!} (n_i+1)^{-m-1} -1\right] f^{(m)}(x)$$

Applying lemma 4.3.2,

$$S_2(x) = \sum_{i=0}^{r-1} C(i,r) \sum_{j=1}^{m} b_j \, (n_i+1)^{-j} f^{(m)}(x)$$

$$= f^{(m)}(x) \sum_{j=1}^{m} b_j \sum_{k=j}^{\infty} a_k \sum_{i=0}^{r-1} C(i,r) \, n_i^{-k}$$

The last equality is a consequence of the fact that

(5.4.2) $$\frac{1}{(n+1)^p} = \sum_{k=p}^{\infty} a_k . n^{-k}.$$

where a_k's are constants independent of n.

then

$$\|f^{(k)} - P_n^{(k)}\|_{[\alpha,\beta]} \le A_r \; E_{n-k}(f^{(k)}) \;, \qquad n \ge k$$

$$\le B_r \; n^{k-r} E_{n-r}(f^{(r)}) \;, \qquad n \ge r$$

where C_r, K_r depend only on r , A_r, B_r depend only on α,β and r and

$$E_n(f) = \inf_{f \in \Pi_n} \|f-p\| \;,$$

Π_n being the set of all algebraic polynomials of degree $\le n$."

Gonska[32] has obtained certain quantitative Korovkin type theorems for simultaneous approximation. As far as simultaneous approximation by linear combinations of linear positive operators is concerned we have the works of Rathore and Singh[68] and Singh[82] on post Widder operators and those of Rathore and Agarwal[67] and Agarwal[1] on exponential type operators. Rathore and Agarwal[67] have obtained:

<u>Theorem</u> <u>1.4.1</u>[67]:(Inverse theorem)- Let $0 < \alpha < 2$, $\lambda_{n+1}/\lambda_n \le C$, $n \in \mathbb{N}$ and for $f \in C(A,B)$,

$$S_\lambda(f,t) = \int_A^B W(\lambda,t,u) \; f(u)du$$

where $W(\lambda,t,u) \ge 0$, $S_\lambda(1,t) \equiv 1$

$$\frac{\partial}{\partial t} W(\lambda,t,u) = \frac{\lambda}{p(t)} \; (u-t) \; W(\lambda,t,u) \;, \quad u,t \in (A,B) \;,$$

$p(t)$ being a polynomial of degree ≤ 2 and positive on (A,B). Further, let $S_\lambda(f,k,t)$ be May[55] type linear combinations of $S_\lambda(f,t)$. Then, (i) $\rightarrow$ (ii) $\leftrightarrow$ (iii) $\rightarrow$ (iv) where

$$\le M\ (n+1)^{-r-\alpha}\ \left[\frac{2}{A}(n+1)X\right]^{\alpha} |f^{(m+l)}(x)|$$

$$\le M\ n^{-r}X^{\alpha}|f^{(m+l)}(x)|$$

writing $l = 2r-k$ and $\alpha = \mu+r-k$ where $k = r+\mu-\alpha$ and using $X \le 1/4$ for all $x \in [0,1]$, the right hand side can be shown not to exceed

$$M\ n^{-r}X^{r-k}|f^{(m+2r-k)}(x)|.$$

Moreover $l+\mu = r+\alpha$ and $r+1 \le l \le 2r-1$ imply $1 \le k \le r-1$, thus by lemma 5.1.3,

$$(5.4.10)\qquad \|S(n,m,l,\mu,0;f,\cdot)\|_{L_p[\frac{A}{n+1},1-\frac{A}{n+1}]} \le M\ n^{-r}\ \|f^{(m)}\|_{p,2r}$$

with this the remaining terms corresponding to $j = 0$ are also taken care of.

So far (5.4.6) has been established for $j = 0,\ 1$ and for all possible values of l and μ. Note that for $l=1$, j is either 0 or 1, so for further analysis let $l \ge 2$.

By (4.1.3), (5.4.2) and (1.5.1d),

$$(5.4.11)\qquad S(n,m,l,\mu,j;f,x) = f^{(m+l)}(x)\sum_{i=0}^{r-1} C(i,r) \sum_{k=1}^{[j/2]} a_{k,j}(x)\ X^k \sum_{q=0}^{k} \binom{k}{q} (-m-1)^{k-q}\ (n_i+1)^{-l-\mu+q}$$

$$= \sum_{k=1}^{[j/2]} a_{k,j}(x) \sum_{q=0}^{k} \binom{k}{q}(-m-1)^{k-q}\ X^k f^{(m+l)}(x) \sum_{\alpha=r}^{\infty} b_\alpha \sum_{i=0}^{r-1} C(i,r)\ n_i^{-\alpha}$$

i) $f^{(m)}$ exists on $[a_1,b_1]$ and

$$\sup_{a_1\le t\le b_1} |S_{\lambda_n}^{(m)}(f,k,t)-f^{(m)}(t)| = O(\lambda_n^{-\alpha(k+1)/2})$$

ii) $f^{(m)} \in Liz(\alpha,k+1;a_2,b_2)$;

iii) (a) If $p< \alpha(k+1) < p+1$, $p=0,1,2,\dots,2k+1$, then

$f^{(p+m)}$ exists and $\in Lip(\alpha(k+1)-p;a_2,b_2)$,

(b) If $\alpha(k+1) = p+1$, $p=0,1,\dots,2k$, then $f^{(p+m)}$ exists and belongs to $Lip^*(1;a_2,b_2)$

iv) $\|S_\lambda^{(m)}(f,k,\cdot) - f^{(m)}\|_{C[a_2,b_2]} = O(\lambda^{-\alpha(k+1)/2})$.

Here the classes $Liz(\alpha,k;a,b)$, $Lip(\alpha;a,b)$ and $Lip^*(1;a,b)$ are defined as follows

$$Liz(\alpha,k;a,b) = \left\{ f\in C[a,b]: \omega_{2k}(f;\delta) = O(\delta^{\alpha k}),\ \delta \to 0 \right\}$$

$$Lip(\alpha;a,b) = \left\{ f\in C[a,b]: \omega_1(f;\delta) = O(\delta^{\alpha}),\ \delta \to 0 \right\} \quad \text{and}$$

$$Lip^*(1;a,b) = \left\{ f\in C[a,b]: \omega_2(f;\delta) = O(\delta),\ \delta \to 0 \right\}$$

where for $m = 1,2,\dots,$ the modulus of continuity $\omega_m(f;\delta) = \omega_m(f,a,b;\delta)$ is given by

$$\omega_m(f;\delta) = \sup_{|h|\le\delta} \left\{ \Big| \sum_{j=0}^{m} (-1)^{m-j}\binom{m}{j} f(x+jh)\Big| : x, x+mh \in [a,b]\right\}$$

The operators $S_\lambda(f,t)$ are said to be regular

If $W(\lambda,t,u)$ regarded as a function in t and u is measurable on $(A,B)\times(A,B)$ and

$$\leq M \sum_{s=0}^{m+1} \sum_{i=0}^{r-1} (n_i+1) \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x)$$

$$\int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} |t-x|^r \frac{1}{|t-x|} \left|\int_x^t |f^{(m+r)}(u)|du\right| dt$$

(5.4.24) $$\leq M\, G(x) \sum_{s=0}^{m+1} \sum_{i=0}^{r-1} R(s,i,x)$$

where $G(x)$ is the Hardy-Littlewood majorant of $f^{(m+r)}(x)$ and

$$R(s,i,x) = (n_i+1) \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x) \int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} |t-x|^r dt.$$

In what follows it is shown that,

(5.4.25) $$|R(s,i,x)| \leq M\, n^{-r} \quad , \qquad \min(x,1-x) \leq \frac{A}{n+1} .$$

By lemma 4.3.3,

$$|R(s,i,x)| \leq (n_i+1)^{-r} \Big(\sum_{\nu=0}^{2r} \sum_{j=0}^{2r-\nu} |T_{n_i-m,j}(x)| \Big)^{1/2} .$$

Since $T_{n_i-m,0}(x) \equiv 1$, $T_{n_i-m,1}(x) \equiv 0$ and for $j \geq 2$ by (4.1.3)

$$|T_{n_i-m,j}(x)| \leq \sum_{l=1}^{[j/2]} |a_{l,j}(x)|((n_i-m)X)^l$$

$$\leq M \sum_{l=1}^{[j/2]} (nX)^l$$

$$\le \int_{\frac{1}{n_i+1}}^{1-\frac{1}{n_i+1}} K(n_i,m,s,x,t)|t-x|^{2r-1} \left|\int_x^t \frac{U^{r-\frac{1}{2}}|f^{(m+2r)}(u)|}{U^{r-\frac{1}{2}}} du\right| dt$$

$$\le \int_{\frac{1}{n_i+1}}^{1-\frac{1}{n_i+1}} K(n_i,m,s,x,t)|t-x|^{2r-1} \left(X^{r-\frac{1}{2}} + T^{r-\frac{1}{2}} \right)$$

$$\left|\int_x^t U^{r-\frac{1}{2}}|f^{(m+2r)}(u)|du\right| dt.$$

Let p = integral part of $\left[\sup_{\substack{\frac{A}{n+1} < x < 1-\frac{A}{n+1} \\ 0 \le i \le r-1}} \left(\frac{n}{X}\right)^{\frac{1}{2}} \left(1-\frac{A}{n+1}-\frac{1}{n_i+1}\right) \right]$

this ensures that $(p+1)\left(\frac{X}{n}\right)^{1/2} \ge \left(1-\frac{A}{n+1}-\frac{1}{n_i+1}\right)$ .So for $\frac{A}{n+1} <$ $x < 1-\frac{A}{n+1}$ and $\frac{1}{n_i+1} < t < 1-\frac{1}{n_i+1}$,

$$\left|\int_x^t du\right| \le \sum_{j=0}^{p} \left(\int_x^{x+(j+1)\left(\frac{X}{n}\right)^{1/2}} + \int_{x-(j+1)\left(\frac{X}{n}\right)^{1/2}}^{x} \right) du$$

$$= \sum_{j=0}^{p} \int_{x-(j+1)\left(\frac{X}{n}\right)^{1/2}}^{x+(j+1)\left(\frac{X}{n}\right)^{1/2}} du$$

$$= \sum_{j=0}^{p} \int_{\frac{A}{n+1}}^{1-\frac{A}{n+1}} \chi_{x,j+1}(u)\, du$$

$$\int_A^B W(\lambda,t,u)dt = a(\lambda), \quad u \in (A,B),$$

where $a(\lambda)$ is a rational function of λ with $a(\lambda)\longrightarrow 1$ as $\lambda\longrightarrow\infty$ and for each fixed $u \in (A,B)$, $m \in \mathbb{N}_o$ and for all λ sufficiently large,

$$t^m p(t)W(\lambda,t,u) \longrightarrow 0 \text{ as } t \longrightarrow A,B$$

Theorem 1.4.2[67](Saturation theorem): If S_λ are regular and $f \in C[a,b]$, then (i) $\to$ (ii) $\to$ (iii) and (iv) $\to$ (v) $\to$ (vi) where

i) $f^{(m)}$ exists on $[a_1,b_1]$ and

$$\lambda_n^{k+1} \sup_{a_1 \le t \le b_1} |S_{\lambda_n}^{(m)}(f,k,t)-f^{(m)}(t)| = O(1)$$

ii) $f^{(2k+m+1)} \in A.C.[a_2,b_2]$ and $f^{(2k+m+2)} \in L_\infty[a_2,b_2]$

iii) $\lambda^{k+1}\|S_\lambda^{(m)}(f,k,\cdot) - f^{(m)}\|_{C[a_3,b_3]} = O(1)$.

iv) $f^{(m)}$ exists on $[a_1,b_1]$ and

$$\lambda_n^{k+1} \sup_{a_1 \le t \le b_1} |S_{\lambda_n}^{(m)}(f,k,t)-f^{(m)}(t)| = o(1)$$

v) $f \in C^{2k+m+2}[a_2,b_2]$ and $\sum_{i=m}^{2k+m+2} Q(i,k,m,t)f^{(i)}(t) = 0$,

where $Q(i,k,m,t)$ are certain polynomials in $t \in [a_2,b_2]$

vi) $\lambda^{k+1}\|S_\lambda^{(m)}(f,k,\cdot) - f^{(m)}\|_{C[a_3,b_3]} = o(1)$.

In the more recent past simultaneous approximation by a variety of operators such as Szász operators, Baskakov-Durrmeyer operators, modified Bernstein polynomials, modified Lupas

Corollary 5.6.1: Let $r,\ m \in \mathbb{N}$ and $f \in W^{m}_{\infty,2r}$, then

$\|P_{n,r}f-f\|_{\infty,p} \leq M\ n^{-r}\|f\|_{m,\infty,2r}$.

Proof: For $r \in \mathbb{N}$ and $f \in L_{\infty,2r}$ we have from [92]

$$\|P_{n,r}f-f\|_{\infty} \leq M\ n^{-r}\|f\|_{\infty,2r} .$$

Using lemma 5.6.1 and (5.6.3) together with this we get the corollary . ■

5.7 INVERSE THEOREM

Throughout thus section the symbols $C_1, C_2, \ldots$ are constants independent of t and n .

THEOREM 5.7.1: Let $r, m \in \mathbb{N}$ and $\phi \in \Phi_{2r}$. Then the following are equivalent

(i) $\quad f \in W^{m,\phi}_{p,2r}$

(ii) $\quad \|P_{n,r}f-f\|_{m,p} = O(\phi(n^{-1/2}))$

(iii) $\quad \|P^{(m)}_{n,r}f-f^{(m)}\|_{p} = O(\phi(n^{-1/2}))$

(iv) $\quad K(t^{2r},f^{(m)}) = O(\phi(t))$

where $K(t,f) = \inf_{g \in L_{p,2r}} \left\{ \|f-g\|_p + t\ \|g\|_{p,2r} \right\}$

Proof: (i) implies (ii)

Corollaries 5.4.1, 5.5.1 and 5.6.1 together give that for $f \in W^{m}_{p,2r}$, $1 \leq p \leq \infty$

(5.7.1) $\quad \|P_{n,r}f-f\|_{m,p} \leq M\ n^{-r}\|f\|_{m,p,2r}$,

operator, modified Szász operators have been studied by Xie Hua Sun[96], Heilmann and Müller[39], Agrawal and Kasana[2], Singh and Prasad[80], Sahai and Prasad[71], Singh[79], Singh, Varshney and Prasad[81] and others.

1.5 DEFINITIONS AND NOTATIONS

In this section we give some basic definitions and notations which we use throughout the thesis.

The symbols $\mathbb{R}, \mathbb{R}^+, \mathbb{N}, \mathbb{N}_o$ and $[x]$ denote the set of real numbers, positive real numbers, positive integers, non-negative integers and integral part of x respectively.

Throughout the thesis, the symbol M has been used as a generic constant independent of f (the function under consideration) and n (as in S_n, B_n and P_n).

For any segment I of $\mathbb{R}$, $C(I)$ and $C^k(I)$ are the spaces of continuous functions and that of k times continuously differentiable functions on I respectively. The class A.C.[a,b] is the collection of absolutely continuous functions on [a,b] and

$$\text{loc A.C.}(I) = \left\{ f \in \text{A.C.}(I') : I' \text{ is a compact sub-interval of } I \right\}$$

Let $1 \le p < \infty$ and I be an arbitrary interval of $\mathbb{R}$.

$$L_p(I) = \left\{ f \text{ measurable on } I : \int_I |f(x)|^p dx < \infty \right\} \quad \text{and}$$

$$\|f\|_{L_p(I)} = \|f\|_p = \left(\int_I |f(x)|^p dx \right)^{\frac{1}{p}}$$

$$L_\infty(I) = \left\{ f \text{ measurable on } I : \operatorname{ess.sup}_I |f(x)| < \infty \right\}$$

$$\|f\|_{L_\infty(I)} = \|f\|_\infty = \operatorname{ess.sup}_I |f(x)|$$

<u>Definition</u> <u>1.5.1</u>: Let $1 \le p < \infty$ and I be an arbitrary interval of $\mathbb{R}$. For $m \in \mathbb{N}$ let

$$U_p^m(I) = \left\{ f \in L_p(I) : f^{(m-1)} \in \text{loc A.C.}(I) \text{ and } f^{(m)} \in L_p(I) \right\}$$

and $\|f\|_{m,p} = \|f\|_p + \|f^{(m)}\|_p$.

<u>Definition</u> <u>1.5.2</u>: For $f \in L_p[a,b]$, $1<p<\infty$, the Hardy-Littlewood majorant H_f of f is defined as follows

$$H_f(x) = \sup_{\substack{t \ne x \\ a \le t \le b}} \frac{1}{|t-x|} \left| \int_x^t |f(u)|du \right|$$

<u>Definition</u> <u>1.5.3</u>[65]:Given a sequence $\{L_n(\cdot,t)\}$, $n \in \mathbb{N}$, of linear positive operators, the linear combinations $L_n(\cdot,r,t)$, $r \in \mathbb{N}$, is defined as follows:

Let $d_0,\dots,d_r$ be $r+1$ distinct positive integers, then

$$L_n(\cdot,r,t) = \frac{1}{\Delta_r} \begin{vmatrix} L_{d_0 n}(\cdot,t) & d_0^{-1} & \cdot & \cdot & \cdot & d_0^{-r} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ L_{d_r n}(\cdot,t) & d_r^{-1} & \cdot & \cdot & \cdot & d_r^{-r} \end{vmatrix}$$

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where Δ_r is the determinant obtained from the displayed determinant after replacing the entries of the first column by 1. This can be shown to be equivalent to the linear combinations used by May[55], defined as follows

Let $d_o, d_1, \dots, d_r$ be $r+1$ arbitrary but fixed positive integers. The operator $S_\lambda(f,r,t)$ is a linear combination of $S_{d_j\lambda}(f,t)$, given by

$$S_\lambda(f,r,t) = \sum_{j=0}^{r} c(j,r)\, S_{d_j\lambda}(f,t)$$

where

$$c(j,r) = \prod_{\substack{i=0 \\ i\neq j}}^{r} \frac{d_j}{d_j - d_i} \quad , \; r \neq 0 \;, \text{ and } c(0,0) = 1 .$$

The linear combinations used in the thesis are a generalization of the above.

<u>Definition</u> <u>1.5.4</u>[25]: Let $n, r \in \mathbb{N}$ and $\{L_n(\cdot,t)\}$ be a sequence of linear positive operators define

$$L_{n,r}(\cdot,t) = \sum_{i=0}^{r-1} C(i,r)\, L_{n_i}(\cdot,t)$$

where $C(i,r)$ and n_i satisfy

(1.5.1a) i) $n = n_o < n_1 < \dots < n_{r-1} < C.n$, $C > 1$ is independent of n.

(1.5.1b) ii) $$\sum_{i=0}^{r-1} C(i,r) = 1$$

(1.5.1c) iii) $$\sum_{i=0}^{r-1} |C(i,r)| < K \qquad (\text{ K independent of } n \text{ })$$

(1.5.1d) iv) $\sum_{i=0}^{r-1} C(i,r)\, n_i^{-s} = 0 \quad , \quad s = 1,2,\ldots,r-1$

The next definition is that of a maximal operator.

Definition 1.5.5: Let τ be a differential expression of the form $\tau = \sum_{k=0}^{n} a_k D^k$, where each a_k is a complex valued function on an arbitrary interval I of $\mathbb{R}$ and $D = \frac{d}{dx}$. For each $n \in \mathbb{N}$ define $A_n(I)$ to be the set of all complex valued functions f on I for which $f^{(n-1)}$ exists and is absolutely continuous on every compact sub-interval of I. Let $A_o(I) \equiv C(I)$. The maximal operator

$T_{\tau,p,q}$ corresponding to τ is defined as follows

The domain of the operator $T_{\tau,p,q}$ is given by

$$D(T_{\tau,p,q}) = \left\{ f : f \in A_n(I) \cap L_p(I) \ , \ \tau f \in L_q(I) \right\} \text{ and}$$

$$T_{\tau,p,q} f = \tau f = \sum_{k=0}^{n} a_k D^k f \quad , \ f \in D(T_{\tau,p,q}) \ .$$

1.6 SOME GENERAL RESULTS

The following result is about certain interpolation inequalities. These inequalities provide a bound for the norm of an intermediate derivative in terms of the norms of the function and its higher derivatives.

Lemma 1.6.1[31]: Let $f \in W_p^m(I)$, $1 \le p \le \infty$ and $m \in \mathbb{N}$. Then,

i) All the derivatives $f^{(k)}$, $0 \le k \le m$, are in $L_p(I)$

ii) For each $\epsilon > o$, there exists a constant K depending only on ϵ,p and the length of the interval I(I may be unbounded), such that for all $f \in W_p^m(I)$

$$\|f^{(k)}\|_p \le K \|f\|_p + \epsilon \|f^{(m)}\|_p$$

The next result is about the Hardy-LIttlewood Majorant of a function $f \in L_p[a,b]$, $1<p<\infty$.

Lemma 1.6.2: Let $f \in L_p[a,b]$, $1 < p < \infty$ and H_f be its Hardy-Littlewood majorant, then

$$\|H_f\|_{L_p[a,b]} \le 2^{1/p} \frac{p}{p-1} \|f\|_{L_p[a,b]}$$

The above lemma follows from [103, pp32] and [84, pp5].

The following generalization of the Berens and Lorentz lemma[15] is due to Avadhani[7].

Lemma 1.6.3[7]: Let $c>0$, Ω be an increasing function on $(0,c]$ and for some $r\in\mathbb{N}$ let $\phi \in \Phi_{2r}$ be such that for all $h,t \in (0,c]$

$$\Omega(h) \le M \left(\phi(t) + \left(\frac{h}{t}\right)^{2r} \Omega(t)\right) .$$

Then

$$\Omega(h) = O(\phi(h)).$$

Lemma 1.6.4[31]: Let I be an arbitrary interval and τ be the differential expression

$$\tau = \sum_{k=0}^{n} a_k D^k \text{ , } a_k \text{ locally integrable on I } 0\le k\le n-1 \text{ and}$$

$$\frac{1}{a_n} \text{ be locally an } L_\infty \text{ function on I.}$$

Then the maximal operator $T_{\tau,p,q}$ $1 \le p,q \le \infty$ is closed.

1.7 THE POLYNOMIAL OPERATORS S_n, B_n AND P_n

Bernstein Polynomials: Ever since their origin in the work of S.N.Bernstein, these polynomials have been studied quite extensively e.g. Becker[9], Becker, Fink and Nessel[10], Berens and Lorentz[15], Butzer[20], Chen[21], Cheng[23], Ditzian[25,26,27], Ivanov[41], de Leeuw[46], Lorentz[49,50], May[55], Miccheli[58], Zhou[102] etc.

Berens and Lorentz[15] proved global inverse theorem for Bernstein polynomials, they established that

"For $f \in C[0,1]$ and $0 < \alpha < 2$, the following statements are equivalent

i) $|B_n f(x)-f(x)| \le K_f \left(\frac{X}{n}\right)^{\alpha/2}$

ii) $f \in Lip^*(\alpha;C[0,1])$

iii) for $0 < \alpha < 1$: $f \in Lip(\alpha;C[0,1])$

for $\alpha=1$: $f \in Lip^*(1;C[0,1])$

for $1 < \alpha < 2$: $f \in C^1[0,1]$ and $f' \in Lip(\alpha-1;C[0,1])$

where Lip and Lip^* are Lipschitz and Zygmund classes respectively and $X=x(1-x)$."

Becker, Fink and Nessel[10] studied approximation by these polynomials in weighted BV spaces. Butzer[20] considered certain linear combinations of these polynomials, he obtained the following result

"If $f(x)$ is defined on $[0,1]$ with $|f(x)| \le K$ and if $f^{(2k)}(x)$ exists at the point x, then

$$|\mathfrak{L}_n^{(2k-2)}(x) - f(x)| = O(n^{-k})$$

moreover

$$|\mathcal{B}_n^{(2k)}(x) - f(x)| = o(n^{-k}) \quad \text{as } n \longrightarrow \infty$$

where $\mathcal{B}_n^{(0)}(x) = B_n f(x)$ and

$$(2^k-1)\mathcal{B}_n^{(2k)}(x) = 2^k \mathcal{B}_{2n}^{(2k-2)}(x) - \mathcal{B}_n^{(2k-2)}(x).$$

He further showed that

If $f^{(2k)}(x)$ exists and is continuous on $[0,1]$ having a modulus of continuity $\omega_{2k}(\delta)$, then

$$|\mathcal{B}_n^{(2k)}(x) - f(x)| \le \max\left\{ \frac{C}{n^k}\,\omega_{2k}(n^{-1/2}),\ \frac{C'}{n^{k+1}} \right\}$$

where $C = C(k)$ and $C' = C'(k,f)$."

Local inverse and saturation theorems for linear combinations of Bernstein polynomials were obtained by May[55]. He considered a more general linear combinations and established his results for a wider class of operators, which are termed as exponential type operators, he proved the following

<u>Theorem</u> <u>1.7.1</u>[55]:(Inverse theorem)- Let $0 < \alpha < 2$, $\lambda_{n+1}/\lambda_n \le C$, $n \in \mathbb{N}$ and $a_1 < a_2 < a_3 < b_3 < b_2 < b_1$.

Then (i)$\longrightarrow$ (ii) $\longrightarrow$ (iii)$\longrightarrow$ (iv) where

i) $\sup\limits_{a_1 \le t \le b_1} |S_{\lambda_n}(f,k,t) - f(t)| = O(\lambda_n^{-\alpha(k+1)/2})$

ii) $f \in Liz(\alpha, k+1; a_2, b_2)$;

iii) (a) If $p < \alpha(k+1) < p+1$, $p = 0,1,2,\dots,2k+1$, then

$f^{(p)}$ exists and $\in Lip(\alpha(k+1)-p; a_2, b_2)$,

(b) If $\alpha(k+1) = p+1$, $p = 0,1,\dots,2k$, then $f^{(p)}$ exists and belongs to $Lip^*(1; a_2, b_2)$

iv) $\|S_{\lambda_n}(f,k,\cdot) - f\|_{C[a_s,b_s]} = O(\lambda^{-\alpha(k+1)/2})$.

When $\alpha = 2$ we have the following

<u>Theorem 1.7.1</u>[55](Saturation theorem): If S_λ are regular as well, then (i) → (ii) → (iii) and (iv) → (v) → (vi) where

i) $$\lambda_n^{k+1} \sup_{a_1 \le t \le b_1} |S_{\lambda_n}(f,k,t)-f(t)| = O(1)$$

ii) $f^{(2k+1)} \in A.C.[a_2,b_2]$ and $f^{(2k+2)} \in L_\infty[a_2,b_2]$

iii) $\lambda^{k+1}\|S_\lambda(f,k,\cdot) - f\|_{C[a_s,b_s]} = O(1)$.

iv) $$\lambda_n^{k+1} \sup_{a_1 \le t \le b_1} |S_{\lambda_n}(f,k,t)-f(t)| = o(1)$$

v) $f \in C^{2k+2}[a_2,b_2]$ and $\sum_{i=k+1}^{2k+2} Q(i,k,t)f^{(i)}(t) = 0$,

where $Q(i,k,t)$are certain polynomials in $t\in[a_2,b_2]$

vi) $\lambda^{k+1}\|S_\lambda(f,k,\cdot) - f\|_{C[a_s,b_s]} = o(1)$.

Ditzian[25] proved a global inverse theorem for the linear combinations of Bernstein polynomials, he established the following

"For $f \in C[0,1]$ and $0 < \beta < 2r$ the following are equivalent

i) $\|B_{n,r}f - f\| = O(n^{-\beta/2})$, $n \longrightarrow \infty$;

ii) $f \in (C,A_{2r})_\beta$;

iii) $\sup_{hr<x<1-hr}|X^{\beta/2}h^{-\beta}\Delta_h^{2r}f(x)| < K$.

where $\Delta_h f(x) = f(x+h/2) - f(x-h/2)$, $X = x(1-x)$, $C \equiv C[0,1]$,

$$A_{2r} = \left\{ f : f^{(2r-1)} \in \text{loc A.C.}(0,1) \text{ and } \sup_{0\le x\le 1} |X^r f^{(2r)}(x)| < \infty \right\}$$

and $(C,A_{2r})_\beta = \left\{ f : K(t^{2r},f) = O(t^\beta) \right\}$, where

$K(t,f) = \inf_{g\in A_{2r}} \left\{ \|f-g\| + t\,\|g\|_{2r} \right\}$, $\|f\| = \sup_{0\le x\le 1} |f(x)|$ and

$\|f\|_{2r} = \sup_{0\le x\le 1} |X^r f^{(2r)}(x)|$."

As for simultaneous approximation by linear combinations of Bernstein polynomials we have the local inverse theorem 1.4.1 of Rathore and Agrawal[67].

Szasz-Mirakjan-Hille operators: These are a generalization of Bernstein polynomials to the infinite interval $[0,\infty)$. Becker[8] obtained the following global inverse theorem for these operators

"For $N \in \mathbb{N}_o$, let $w_o(x) = 1$, $w_N(x) = (1+x^N)^{-1}$ and

$C_N = \left\{ f\in C[0,\infty) : w_N f \text{ is uniformly continuous and bounded on } (0,\infty) \right\}$,

then for $f \in C_N$ and $0 < \alpha \le 2$

$$w_N(x)|S_n f(x) - f(x)| \le K_N \left(\frac{x}{n}\right)^{\alpha/2}, \quad n\in\mathbb{N} \text{ and } x\in[0,\infty),$$ is

equivalent to $f \in \mathrm{Lip}_N^2\alpha = \left\{ f\in C_N : \sup_{0<h\le\delta} \|\Delta_h^2 f\|_N = O(\delta^\alpha),\ \delta \to 0 \right\}$

where $\|f\|_N = \sup_{x\ge 0} w_N(x)|f(x)|$."

As for approximation by linear combinations of these operators we have the following results due to Rathore[65]

"If $f^{(2k)}(x)$ exists at the point x, then

$$|\Xi_n^{[2k-2]}(x) - f(x)| = O(n^{-k})$$

and

$$|\Xi_n^{(2k)}(x) - f(x)| = o(n^{-k}) \text{ as } n \longrightarrow \infty$$

He further showed that

If $f^{(2k)}(x)$ exists and is continuous on $\langle a,b\rangle$, then

$$|\Xi_n^{(2k)}(x) - f(x)| \leq \max\left\{ \frac{C}{n^k}\,\omega_{2k}(n^{-1/2}),\ \frac{C'}{n^{k+1}} \right\}$$

where $C = C(k)$ and $C' = C'(k,f)$, $x \in [a,b]$ and $\langle a,b\rangle$ denotes some open neighborhood of $[a,b]$ with $a \geq 0$."

The combinations are same as those of Butzer[20].

May[55] proved local inverse and saturation theorem for linear combinations of these operators and Rathore and Agrawal[67] obtained the same for simultaneous approximation. These results have already been mentioned.

Szász operators have been studied by various others including Amanov[6], Cheng[22], Pethe[64], Xie Hua Sun[96], Zhou[99,100] and others.

Bernstein-Kantorovitch polynomials: Bernstein polynomials are not bounded operators on $L_p[0,1]$. Kantorovitch[42] gave the following modification of these polynomials

$$P_n f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}\, (n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt\ .$$

P_n turns out to be bounded operator on $L_p[0,1]$, $1 \leq p \leq \infty$. A brief survey of some of the works in the literature for Bernstein-Kantorovitch polynomials is being given below.

L_p-approximation by Bernstein-Kantorovitch polynomials has been studied by Hoeffding[40], Bojanic and Shisha[18], Becker

and Nessel[12,14], Becker, Lautner, Nessel and Worms[11], Grundmann[34,35], Butzer[19], Ditzian[26], Ditzian and May[29], Maier[53], May[56], Müller[61],, Nagel[62], Reimenschneider[69], Sinha[83], Totik[86,87,88], Winslin[92], Agrawal and Prasad[4], Zhou[101] etc.

Ditzian and May[29] have proved direct, inverse and saturation theorems for these polynomials. The inverse theorem is as follows

Let $0 < \alpha < 2$, $1 \le p < \infty$ and $f \in L_p[0,1]$. Then for $a<a_1<b_1<b$, $\|P_n f-f\|_{L_p[a,b]} = O(n^{-\alpha/2})$, $n \longrightarrow \infty$, implies that $\omega_2(f,h)_{L_p[a_1,b_1]} = O(h^{\alpha})$, $h \longrightarrow 0$. when $\alpha = 2$, following is the saturation theorem

$\|P_n f-f\|_{L_p[a,b]} = O(n^{-1})$, $n \longrightarrow \infty$, implies that f coincides a.e. on $[a,b]$ with a function F such that $F' \in A.C.[a,b]$ and $F^{(2)} \in L_p[a,b]$ for $p > 1$ and $F' \in BV[a,b]$ for $p=1$ and

$\|P_n f-f\|_{L_p[a,b]} = O(n^{-1})$, $n \longrightarrow \infty$, implies that f coincides a.e. on $[a,b]$ with a function F satisfying for some constant C

$t(1-t)\ F'(t) = C$ for $t \in [a,b]$.

Bounds for error in L_p-approximation for p=1 and p>1 has been studied by Grundmann[34,35] and Müller[61] respectively. Bojanic and Shisha obtained similar estimates in weighted L_1-norm. For Differentiable functions bounds for error in approximation has been given by Hoeffding[40] and Müller[61].

Maier[53] proved global approximation theorem in $L_1[0,1]$. He obtained the following

Let $S = \left\{ f \;\middle|\; \begin{array}{l} f(x) = k + \int_{\xi}^{x} \frac{h(t)}{t(1-t)}\,dt,\ \xi \in (0,1),\ k \in \mathbb{R} \\ \text{and } h \in BV[0,1] \text{ satsfying } h(0) = h(1) = 0 \end{array} \right\}$

Then $\|P_n f - f\|_{L_1[0,1]} = O(n^{-1})$ if and only if $f \in S$, and

$\|P_n f - f\|_{L_1[0,1]} = o(n^{-1})$ if and only if f is constant a.e.

This has been extended to $L_p[0,1]$, $1 < p < \infty$, by Reimenschneider[69]. May[56] proved global saturation theorem along with a correction term in weighted $L_p[0,1]$ norm, where $1 \le p < \infty$. Becker and Nessel[12] have characterized the saturation class of the Bernstein Kantorovitch polynomials.

Totik[87,88] has obtained the following inverse theorems

"If $1 \le p < \infty$, $0 < \alpha < 1$ and $f \in L_p(0,1)$ then

i) $\|P_n f - f\|_p = O(n^{-\alpha})$ and

ii) a. $\|\Delta^{*}_{h\sqrt{x(1-x)}}(f;x)\|_{L_p(h^2,1-h^2)} = O(h^{2\alpha})$,

b. $\|f(\cdot+h) - f(\cdot)\|_{L_p(0,1-h)} = O(h^{\alpha})$

are equivalent."

Here $\Delta^{*}_{h}(f;x) = f(x-h) - 2f(x) + f(x+h)$. He later showed that (b) is not required in fact (b) is implied by (a).

Sinha[83] proved local direct, inverse and saturation theorem for linear combinations of Bernstein kantorovitch polynomials.

Let $I = [0,1]$, $I_j = [a_j, b_j]$, $j=1,2,3$ where $0<a_j<a_{j+1}$ and $b_{j+1}<b_j<1$ and $P_n(f,k,t)$ is a May[55] type linear

combinations of Bernstein Kantorovitch polynomials.

i) Let $1<p<\infty$ and $f \in L_p(I)$. If f has $(2k+2)$ derivatives on I_1 with $f^{(2k+1)} \in \text{loc A.C.}(I_1)$ and $f^{(2k+2)} \in L_p(I_1)$ then

$$\|P_n(f,k,\cdot)-f\|_{L_p(I_2)} \le \frac{M}{n^{k+1}} \{ \|f^{(2k+2)}\|_{L_p(I_1)} + \|f\|_{L_p(I)} \}$$

ii) Let $f \in L_1(I)$. If f has $2k+1$ derivatives on I_1 with $f^{(2k)} \in A.C.(I_1)$ and $f^{(2k+1)} \in BV(I_1)$, then for some constant M

$$\|P_n(f,k,\cdot)-f\|_{L_1(I_2)} \le \frac{M}{n^{k+1}} \{ \|f^{(2k+1)}\|_{BV(I_1)} + \|f^{(2k+1)}\|_{L_1(I_2)} + \|f\|_{L_1(I)} \}$$

iii) Let $0 < \alpha < 2k+2$, $1 \le p < \infty$ and $f \in L_p(I)$. Then

$$\|P_n(f,k,\cdot)-f\|_{L_p(I_1)} = O(n^{-\alpha/2}), \quad n \longrightarrow \infty$$

implies that

$$\omega_{2k+2}(f,h,p,I_2) = O(h^{\alpha}), \quad h \longrightarrow 0.$$

iv) The following implication (a)$\longrightarrow$(b)$\longrightarrow$(c) and (d)$\longrightarrow$(e)$\longrightarrow$(f) hold

(a) $$\|P_n(f,k,\cdot)-f\|_{L_p(I_1)} = O(n^{-(k+1)})$$

(b) f coincides a.e. with a function F on I_2 having $2k+2$ derivatives such that (1) when $p > 1$, $F^{(2k+1)} \in A.C.(I_2)$ and $F^{(2k+2)} \in L_p(I_2)$ and (2) when $p = 1$, $F^{(2k)} \in A.C.(I_2)$ and $F^{(2k+1)} \in BV(I_2)$.

(c) $$\|P_n(f,k,\cdot)-f\|_{L_p(I_3)} = O(n^{-(k+1)})$$

(d) $$\|P_n(f,k,\cdot)-f\|_{L_p(I_1)} = o(n^{-(k+1)})$$

(e) f coincides a.e. with a function F on I_2, where F is 2k+2 times continuously differentiable on I_2 and satisfies

$$\sum_{j=1}^{2k+2} Q(j,k,t)\ F^{(j)}(t) = 0$$

where Q(j,k,t) are certain polynomials in t.

(f) $$\|P_n(f,k,\cdot)-f\|_{L_p(I_3)} = o(n^{-(k+1)})$$

As for the global inverse theorem for linear combinations of Bernstein Kantorovitch polynomials Winslin[92] proved the following

"Let $0 < \beta < 2r$ and $1 \le p \le \infty$. Then, for $f \in L_p[0,1]$

$$\|P_{n,r}f - f\|_p = O(n^{-\beta/2})$$

if and only if $f \in L^{\beta}_{p,2r}$."

1.8 CONTENTS OF THE THESIS

Apart from Chapter I, the present chapter, the thesis has other four chapters II-V. A chapter wise summary of the remaining chapters is given below.

Chapter II: Ordinary approximation by linear combinations of Szász-Mirakjan-Hille operators is studied in this chapter. In §1 some preliminary results about the operator are given and the spaces $\mathcal{D}_{2r}$ and $\mathcal{D}_{2r,\phi}$ are defined. Certain Bernstein type inequalities are established in §2 and §3.A direct theorem for

linear combinations of $S_n f$'s, $f \in \mathcal{D}_{2r}$, is obtained in §4. The final section of the chapter contains a global inverse theorem of a general order for linear combinations of S_n.

<u>Chapter III</u>: Simultaneous approximation by linear combinations of Szász-Mirakjan-Hille operators have been studied in this chapter In §1 generalizations of the Bernstein type inequalities of the previous chapter are given . §2 contains a direct theorem for simultaneous approximation by linear combinations of the operator.Inverse theorem of a general order for linear combinations of the operator is established in §3.

<u>Chapter IV</u>: concerns with simultaneous approximation by linear combinations of Bernstein polynomials . §1 contains some preliminary results about the Bernstein polynomials and definitions of the spaces A^m_{2r} and $A^m_{2r,\phi}$ .In §2 Bernstein type inequalities for $f \in C^m[0,1]$ and $f \in A^m_{2r}$ are established. A direct theorem for simultaneous approximation by linear combinations has been proved in §3. The final section, §4, contains a global inverse theorem of a general order in simultaneous approximation by linear combinations of B_n.

<u>Chapter V</u>: Here the object of study is simultaneous approximation by linear combinations of Bernstein-Kantorovitch polynomials . §1 contains preliminary results about the operator and the spaces $W^m_{p,2r}$ and $W^{m,\phi}_{p,2r}$, $1 \le p < \infty$. The next two sections, §2 and §3, contain Bernstein type inequalities for f

belonging to $W^m_{p,2r}$ and $W^{m,\phi}_{p,2r}$ respectively. In § 4 a direct theorem for linear combinations of $P_n f$, $f \in W^m_{p,2r}$, $1 < p < \infty$, has been proved. In § 5 the same theorem is established for the case $p = 1$. The case $p = \infty$ is dealt with in §6. The last section contains the inverse theorem.

CHAPTER II

GLOBAL INVERSE THEOREM FOR LINEAR COMBINATION OF SZASZ-MIRAKJAN-HILLE OPERATOR

In this chapter we consider linear combinations of the Szasz-Mirakjan-Hille operator. For $f \in C[0,\infty)$ this is given by

$$S_n(f,x) = S_nf(x) = \sum_{k=0}^{\infty} q_{n,k}(x)\ f(\tfrac{k}{n}), \qquad n \in \mathbb{N},\ k \in \mathbb{N}_o$$

where $q_{n,k}(x) = e^{-nx}\dfrac{(nx)^k}{k!}$

The first section contains some preliminary results about $S_nf(x)$ and $q_{n,k}(x)$ and some definitions. These are used in establishing later results. Bernstein type inequalities for $f \in C[0,\infty)$ and $f \in \mathcal{D}_{2r}$, which will be defined in section one, are proved in second and third sections respectively. A direct theorem for linear combinations of $S_nf(x)$, $f \in \mathcal{D}_{2r}$ is proved in section four. The last section contains global inverse theorem of a general order for linear combinations of $S_nf(x)$

2.1 PRELIMINARIES

The operator S_n from $C[0,\infty)$ to $C[0,\infty)$ is bounded, we have

$$(2.1.1) \qquad |S_nf(x)| \le \sum_{k=0}^{\infty} q_{n,k}(x)|f(\tfrac{k}{n})| \le \|f\|_{\infty}.$$

Next we give two lemmas concerned with the kernel $q_{n,k}(x)$ of the operator S_n.

<u>LEMMA 2.1.1</u>: Let $r \in \mathbb{N}_0$, then

$$q^{(r)}_{n,k}(x) = \sum_{l,m\geq 0} c(r,l,m) \frac{(k-nx)^{r-2l-m}\, n^l}{x^{r-l}} q_{n,k}(x) \tag{2.1.2}$$

where $l,m \in \mathbb{N}_0$, $c(r,l,m)$ is a constant independent of x,n and k and the sum runs over all values of l, m for which $r-2l-m \geq 0$.

<u>Proof</u>: We have

$$\frac{d}{dx} q_{n,k}(x) = q'_{n,k}(x) = n\, k\, e^{-nx} \frac{(nx)^{k-1}}{k!} - n\, e^{-nx} \frac{(nx)^k}{k!}$$

$$= \frac{(k-nx)}{x} q_{n,k}(x)$$

Since $r = 1$, the only possible values of l, $m \geq 0$ such that $r - 2l - m \geq 0$ are $l = 0$ and $m = 0$ and 1. Therefore,

$$q'_{n,k}(x) = \sum_{l,m\geq 0} c(1,l,m) \frac{(k-nx)}{x} q_{n,k}(x)$$

where $c(1,0,0) = 1$ and $c(1,0,1) = 0$.

So the lemma is true for $r = 1$. Suppose it is true up to $r-1$. Assume $q^{(r-1)}_{n,k}(x)$ can be written in the form (2.1.2) where $r-1-2l-m \geq 0$. Therefore,

$$q^{(r)}_{n,k}(x) = \frac{d}{dx}\left[\sum_{l,m\geq 0} c(r-1,l,m) \frac{(k-nx)^{r-1-2l-m}\, n^l}{x^{r-1-l}} q_{n,k}(x)\right]$$

$$= \sum_{l,m\geq 0} c(r-1,l,m) \Bigg[\frac{(k-nx)^{r-2l-m}\, n^l}{x^{r-l}} q_{n,k}(x)$$

$$- (r-1-2l-m) \frac{(k-nx)^{r-2l-m-2}\, n^{l+1}}{x^{r-l-1}} q_{n,k}(x)$$

$$- (r-l-1) \frac{(k-nx)^{r-2l-m-1}\, n^l}{x^{r-l}} q_{n,k}(x) \Bigg],$$

note that for $r-1-2l-m = 0$ the second term within the brackets will not be there and when $r-1-2l-m \geq 1$ then $r-2l-m-2 \geq 0$. So the whole expression can be grouped together and written as in (2.1.2) with the sum over all possible values of l, $m \geq 0$ such that $r-2l-m \geq 0$. This establishes the lemma for r. Hence by the principle of induction the lemma follows for all $r \in \mathbb{N}_o$.∎

<u>LEMMA</u> <u>2.1.2</u>: For $m \in \mathbb{N}_o$ and $x \in [0,\infty)$ we have

$$\sum_{k=0}^{\infty} q_{n,k}(x) \frac{n^m}{(k+1)^m} \leq \frac{m!}{x^m}$$

where $q_{n,k}(x) = e^{-nx} \frac{(nx)^k}{k!}$.

<u>Proof:</u>

$$\frac{n^m}{(k+1)^m} q_{n,k}(x) = \frac{1}{x^m} \frac{1}{(k+1)^m} e^{-nx} \frac{(nx)^{k+m}}{k!}$$

$$= \frac{1}{x^m} \frac{(k+1)\ \ldots\ (k+m)}{(k+1)^m} e^{-nx} \frac{(nx)^{k+m}}{(k+m)!}$$

$$\leq \frac{m!}{x^m} e^{-nx} \frac{(nx)^{k+m}}{(k+m)!}$$

So

$$\sum_{k=0}^{\infty} \frac{n^m}{(K+1)^m} q_{n,k}(x) \leq \frac{m!}{x^m} \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^{k+m}}{(k+m)!} \leq \frac{m!}{x^m}. \blacksquare$$

Following lemma expresses the derivative of $S_n f(x)$ in terms of the forward difference operator Δ.

<u>LEMMA</u> <u>2.1.3</u>[54]: For $f \in C[0,\infty)$ and $m \in \mathbb{N}_o$

$$(S_n f)^{(m)}(x) = S_n^{(m)} f(x) = n^m \sum_{k=0}^{\infty} q_{n,k}(x) \Delta_{1/n}^m f(\tfrac{k}{n})$$

where $\Delta_h f(x) = f(x+h) - f(x)$.

The next lemma is about the moments of S_n,

$$S_n(\cdot - x)^s, x) = \sum_{k=0}^{\infty} q_{n,k}(x) \left(\frac{k}{n} - x\right)^s, \quad s \in \mathbb{N}_o$$

$$= \frac{1}{n^s} \sum_{k=0}^{\infty} q_{n,k}(x) (k-nx)^s$$

<u>LEMMA</u> <u>2.1.4</u>: Let $T_{n,s}(x) = \sum_{k=0}^{\infty} q_{n,k}(x) (k-nx)^s$, $s \in \mathbb{N}_o$, then

$T_{n,0}(x) \equiv 1$, $T_{n,1}(x) \equiv 0$ and for $s \geq 2$

(2.1.4) $$T_{n,s}(x) = \sum_{j=0}^{[s/2]} a_{j,s} (nx)^j$$

where $a_{j,s}$ are constants independent of x and n .

<u>Proof</u>:

(2.1.5) $$T_{n,0}(x) = \sum_{k=0}^{\infty} q_{n,k}(x) = 1$$

and $$T_{n,1}(x) = \sum_{k=0}^{\infty} q_{n,k}(x) (k-nx)$$

$$= \sum_{k=1}^{\infty} \frac{(nx)^k}{(k-1)!} e^{-nx} - nx = 0$$

We have the following recurrence relation [8,65] for $T_{n,s}(x)$

(2.1.6) $$T_{n,s+1}(x) = x(T'_{n,s}(x) + s\, n\, T_{n,s-1}(x))$$

We complete the proof using induction. For s = 2 and 3 we have using (2.1.6),

$$T_{n,2}(x) = nx$$
$$T_{n,3}(x) = nx$$

So (2.1.4) holds good for s = 2 and 3 . Suppose it is true upto s, we will show that it is true for s+1 as well. Substituting for $T_{n,s}(x)$ and $T_{n,s-1}(x)$ in (2.1.6),

$$T_{n,s+1}(x) = x\left[\frac{d}{dx} \left(\sum_{j=1}^{[s/2]} a_{j,s} (nx)^j \right) + s\, n \sum_{j=1}^{[\frac{s-1}{2}]} a_{j,s-1} (nx)^j \right]$$

$$= x \left(\sum_{j=1}^{[s/2]} a_{j,s}\, jn\, (nx)^{j-1} + s\, n \sum_{j=1}^{[\frac{s-1}{2}]} a_{j,s-1} (nx)^j \right)$$

$$= \left(\sum_{j=1}^{[s/2]} a_{j,s}\, j\, (nx)^j + s \sum_{j=1}^{[\frac{s-1}{2}]} a_{j,s-1} (nx)^{j+1} \right)$$

$$= \left(\sum_{j=1}^{[s/2]} a_{j,s}\, j\, (nx)^j + s \sum_{j=2}^{[\frac{s-1}{2}]+1} a_{j-1,s-1} (nx)^j \right)$$

$$= \begin{cases} \displaystyle\sum_{j=1}^{[s/2]} a_{j,s}\, j\, (nx)^j + s \sum_{j=2}^{[\frac{s+1}{2}]} a_{j-1,s-1} (nx)^j & , s \text{ odd} \\ \displaystyle\sum_{j=1}^{[\frac{s+1}{2}]} a_{j,s}\, j\, (nx)^j + s \sum_{j=2}^{[s/2]} a_{j-1,s-1} (nx)^j & , s \text{ even} \end{cases}$$

Therefore suitably redefining the constants $a_{j,s}$ the above can be written as

$$= \sum_{j=1}^{[\frac{s+1}{2}]} a_{j,s+1} (nx)^j$$

This completes induction and hence the lemma.■

As a consequence of this lemma we have a very useful

<u>Corollary</u> <u>2.1.1</u>: For $x \ge A/n$, A some constant, we have

$$|T_{n,s}(x)| \le M\,(nx)^{[s/2]},\ s \ge 2$$

<u>Proof</u>: Since $x \ge A/n$, so $nx/A \ge 1$ always. This together with Lemma 2.1.4 gives us,

$$|T_{n,s}(x)| \le \sum_{j=1}^{[s/2]} |\,a_{j,s}|\,(nx)^{j}\,(\frac{nx}{A})^{[s/2]-j} \quad \le \quad M\,(nx)^{[s/2]} \quad \blacksquare$$

The Spaces $\mathcal{D}_{2r}$ and $\mathcal{D}_{2r,\varphi}$

<u>Definition</u> <u>2.1.1</u>: A function $f \in C[0,\infty)$ is said to belong to $\mathcal{D}_{2r}$ provided $f^{(2r)}$ exists and

a) $f^{(2r-1)}(x) \in$ loc A.C.$(0,\infty)$

b) $\sup\limits_{x\ge0} |x^{i} f^{(r+i)}(x)| < \infty$, $i = 0,\dots,r-1$

c) $\operatorname{ess.sup}\limits_{x\ge0} |x^{r} f^{(2r)}(x)| < \infty$

The space $\mathcal{D}_{2r}$ is normed by the norm

$$\|f\|_{d,2r} = \|f\|_{\infty} + \|f^{(r)}\|_{\infty} + \|x^{r} f^{(2r)}\|_{\infty} \quad ,$$

where

$$\|f\|_{\infty} = \operatorname{ess.sup}\limits_{x\ge0} |f(x)|$$

For $\varphi \in \Phi_{2r}$, the space $\mathcal{D}_{2r,\varphi}$ is given by

$$\left\{f \in C[0,\infty) : K(t^{2r}, f) = O(\varphi(t))\right\}$$

where $K(t,f)$ is the Peetre's K-functional,

$$K(t, f) = \inf_{g\in\mathcal{D}_{2r}} \left\{ \|f-g\|_{\infty} + t\|g\|_{d,2r}\right\}$$

The following lemma is an interpolation inequality. This provides a bound for $\|x^{i} f^{(r+i)}\|_{\infty}$ in terms of $\|f^{(r)}\|_{\infty}$ and $\|x^{r} f^{(2r)}\|_{\infty}$.

<u>LEMMA</u> <u>2.1.5</u>: Let $f \in \mathcal{D}_{2r}$. Then given $\epsilon > 0$ there exists a constant M independent of f such that for $i = 1,\dots,r-1$

$$(2.1.7)\qquad \|x^i f^{(r+i)}(x)\|_\infty \le M \|f^{(r)}\|_\infty + \epsilon \|x^r f^{(2r)}\|_\infty$$

<u>Proof</u>: Let $f \in \mathcal{D}_{2r}$ we define a function F on $(-\infty,\infty)$ by $F(s) = f^{(r)}(e^s)$ Then $F(s) \in C(\mathbb{R})$ and $\|F\|_\infty = \|f^{(r)}\|_\infty$. For $i = 1,\dots,r$, we can find scalars $a_1(i), \dots, a_{i-1}(i)$ such that

$$(2.1.8)\qquad \frac{d^i}{ds^i} F(s) = e^{is} f^{(r+i)}(e^s) + a_{i-1}(i)\, e^{s(i-1)} f^{(r+i-1)}(e^s) + \dots + a_1(i)\, e^s f^{(r+1)}(e^s)$$

Let $x = e^s$, so

$$(2.1.8a)\qquad \frac{d^i}{ds^i} F(s) = x^i f^{(r+i)}(x) + a_{i-1}(i)\, x^{i-1} f^{(r+i-1)}(x) + \dots + a_1(i)\, x\, f^{(r+1)}(x)$$

Consequently we can find scalars $b_1(i), \dots, b_{i-1}(i)$ such that

$$(2.1.9)\qquad x^i f^{(r+i)}(x) = \frac{d^i}{ds^i} F(s) + b_{i-1}(i) \frac{d^{i-1}}{ds^{i-1}} F(s) + \dots + b_1(i) \frac{d}{ds} F(s)$$

and we let $b_i(i) = 1$.

Since $f(x)$, $x^i f^{(r+i)}(x) \in C[0,\infty)$ for $i = 1,\dots,r$, (2.1.8a) implies that $F(s)$ and $F^{(r)}(s) \in C(\mathbb{R})$

Using Lemma 1.6.1, for the given ϵ we can find a constant M_1 depending only on ϵ such that for $i = 1,\dots, r-1$

$$(2.1.10)\qquad \|F^{(i)}\|_\infty \le M_1 \|F\|_\infty + \frac{\epsilon}{2\alpha} \|F^{(r)}\|_\infty$$

where

$$(2.1.11) \qquad \alpha = \sup_{1\le i\le r} \left\{ \sum_{j=1}^{i} |b_j(i)| \right\} < \infty ,$$

since, $b_i(i) = 1$ for all i , $\alpha \neq 0$.

Without loss of generality we can assume

$$(2.1.12) \qquad 1 - \epsilon \sum_{j=1}^{r-1} |b_j(r)| \ > \ \frac{1}{2}$$

Using (2.1.9-10) we get

$$(2.1.13) \qquad \|x^i f^{(r+i)}\|_\infty \le \alpha M_1 \|F\|_\infty + \frac{\epsilon}{2} \|F^{(r)}\|_\infty$$

choosing M_2 such that for $i = 1, \ldots, r-1$

$$\|F^{(i)}\|_\infty \le M_2 \|F\|_\infty + \epsilon \|F^{(r)}\|_\infty$$

and using (2.1.9) for $i = r$

$$\|x^r f^{(2r)}\|_\infty \ge \|F^{(r)}\|_\infty - \sum_{j=1}^{r-1} |b_j(r)| \ (M_2 \|F\|_\infty + \epsilon \|F^{(r)}\|_\infty)$$

$$\ge \|F^{(r)}\|_\infty \ (1 - \epsilon \sum_{j=1}^{r-1} |b_j(r)|) - M_2 \sum_{j=1}^{r-1} |b_j(r)| \ \|F\|_\infty$$

$$(2.1.14) \qquad \ge \frac{1}{2} \|F^{(r)}\|_\infty - M_2 \sum_{j=1}^{r-1} |b_j(r)| \ \|F\|_\infty \qquad \text{(using 2.1.12)}$$

The estimates (2.1.13-14) imply for $i = 1, \ldots, r-1$

$$\|x^i f^{(r+i)}\|_\infty \le \alpha M_1 \|F\|_\infty + \epsilon \|x^r f^{(2r)}\|_\infty$$

$$+ \epsilon M_2 \sum_{j=1}^{r-1} |b_j(r)| \ \|F\|_\infty$$

$$\le M \|F\|_\infty + \epsilon \|x^r f^{(2r)}\|_\infty .$$

$$= M \|f^{(r)}\|_\infty + \epsilon \|x^r f^{(2r)}\|_\infty .$$

This proves our assertion.■

Corollary 2.1.2: For $f \in \mathcal{D}_{2r}$,

$$\|x^i f^{(r+i)}\|_\infty \le M \|f^{(r)}\|_\infty + \|x^r f^{(2r)}\|_\infty \quad , \quad i = 1,\dots,r-1$$

where M is as in Lemma 2.1.5.

THEOREM 2.1.1: The space $\mathcal{D}_{2r}$ is complete with respect to the norm $\|\cdot\|_{d,2r}$.

Proof: For $f \in \mathcal{D}_{2r}$ define $\|f\| = \|f\|_\infty + \sum_{i=r}^{2r} \|x^{i-r} f^{(i)}\|_\infty$. In view of the above corollary we have that $\|\cdot\|$ and $\|\cdot\|_{d,2r}$ are equivalent on $\mathcal{D}_{2r}$. Thus it is sufficient to show that $(\mathcal{D}_{2r}, \|\cdot\|)$ is complete.

Suppose $\{f_n\}$ is a Cauchy sequence in $(\mathcal{D}_{2r}, \|\cdot\|)$, then $\{f_n\}$ and $\{x^{i-r} f_n^{(i)}\}$, $i = r, \dots, 2r$, are cauchy in $(C[0,\infty), \|\cdot\|_\infty)$. But $(C[0,\infty), \|\cdot\|_\infty)$ being complete we have f and g_i, $i = r,\dots,2r$, in $C[0,\infty)$ such that $\|f_n - f\|_\infty$ and $\|x^{i-r} f_n^{(i)} - g_i\|_\infty$, $i = r,\dots,2r$ go to zero as n goes to infinity.

Define

$$\tau = \sum_{i=r}^{2r} x^{i-r} D^i \quad \text{and}$$

$$\tau_j = \sum_{\substack{i=r \\ i \ne j}}^{2r} x^{i-r} D^i - x^{j-r} D^j \quad , \quad j = r,\dots,2r.$$

Following the notations of definition 1.5.5 we have

$$D(T_{\tau,\infty,\infty}) = \left\{ f : f^{(2r-1)} \in \text{loc.A.C.}(0,\infty) \text{ and } \|\tau f\|_\infty < \infty \right\} \text{ and}$$

$$D(T_{\tau_j,\infty,\infty}) = \left\{ f : f^{(2r-1)} \in \text{loc.A.C.}(0,\infty) \text{ and } \|\tau_j f\|_\infty < \infty \right\}$$

clearly $\mathcal{D}_{2r} = \bigcap_{j=r}^{2r} D(T_{\tau_j,\infty,\infty}) \cap D(T_{\tau,\infty,\infty})$.

By lemma 1.6.4, $T_{\tau,\infty,\infty}$ and $T_{\tau_j,\infty,\infty}$, $j = r,...,2r$, are closed

Also $T_{\tau,\infty,\infty}f_n$ converges to $\sum_{i=r}^{2r} g_i$ and $T_{\tau_j,\infty,\infty}f_n$ converges to $\sum_{i=r}^{2r} g_i - g_j$ in the sup norm. Thus the closedness of the operators imply

$$f \in \bigcap_{j=r}^{2r} D(T_{\tau_j,\infty,\infty}) \cap D(T_{\tau,\infty,\infty}) \qquad T_{\tau,\infty,\infty}f = \sum_{i=r}^{2r} g_i \quad \text{and}$$

$$T_{\tau_j,\infty,\infty}f = \sum_{i=r}^{2r} g_i - g_j \;,\; j = r,...,2r.$$

From here it follows that $f \in \mathcal{D}_{2r}$ and $g_i = x^{i-r} f^{(i)}$, $i=r,...,2r$.

Hence the theorem. ■

2.2 BERNSTEIN TYPE INEQUALITY FOR $f \in C[0,\infty)$

<u>LEMMA 2.2.1</u>: For $f \in C\,[0,\infty)$, $r \in \mathbb{N}$,

$$\|x^r S_n^{(2r)} f\|_\infty \le M \; n^r \; \|f\|_\infty$$

<u>Proof:</u> Since $|\Delta_{1/n}^{2r} f(\frac{k}{n})| \le \sum_{j=0}^{2r} \binom{2r}{j} |f(\frac{k+j}{n})| \le 2^{2r} \|f\|_\infty$

By lemma 2.1.3,

$$|x^r \; S_n^{(2r)} f(x)| \le n^{2r}.x^r \sum_{k=0}^{\infty} |\Delta_{1/n}^{2r} f(\tfrac{k}{n})| \; q_{n,k}(x)$$

$$\le (nx)^r \; n^r \; 2^{2r} \; \|f\|_\infty \sum_{k=0}^{\infty} q_{n,k}(x)$$

Therefore,

(2.2.1) $$|x^r S_n^{(2r)} f(x)| \le M\, n^r \|f\|_\infty \quad , \quad x \le 1/n$$

For $x \ge 1/n$ we make use of the lemma 2.1.1 which tells us that $q_{n,k}^{(2r)}(x)$ is a sum of terms of the type

$$c(2r,l,m)\, \frac{(k-nx)^{2r-2l-m}\; n^l}{x^{2r-l}}\, q_{n,k}(x)$$

where $l,m \ge 0$, $2r-2l-m \ge 0$ and $c(2r,l,m)$ is a constant independent of n and k. So $x^r S_n^{(2r)} f(x)$ is a sum of terms of the type

$$S(l,m,n,x) = c(2r,l,m)\, x^r \frac{n^l}{x^{2r-l}} \sum_{k=0}^{\infty} q_{n,k}(x)\; (k-nx)^{2r-2l-m}\; f(\tfrac{k}{n})$$

Thus,

$$|S(l,m,n,x)| \le M\, \frac{n^l}{x^{r-l}} \sum_{k=0}^{\infty} q_{n,k}(x)\; |k-nx|^{2r-2l-m}\; |f(\tfrac{k}{n})|$$

$$\le M\; \|f\|_\infty \frac{n^l}{x^{r-l}} \sum_{k=0}^{\infty} q_{n,k}(x)\; |k-nx|^{2r-2l-m}$$

Since $\sum_{k=0}^{\infty} q_{n,k}(x) = 1$. The right hand side of the above inequality can be shown, using Cauchy Schwarz inequality, not to exceed

$$M\; \|f\|_\infty \frac{n^l}{x^{r-l}} \left(\sum_{k=0}^{\infty} q_{n,k}(x)\; (k-nx)^{4r-4l-2m} \right)^{1/2}$$

Here, the term within the brackets is $T_{n,4r-4l-2m}(x)$. Therefore, corollary 2.1.1 gives

$$|S(l,m,n,x)| \le M\; \|f\|_\infty \frac{n^l}{x^{r-l}} \left[(nx)^{2r-2l-m} \right]^{1/2}$$

$$= M n^r \|f\|_\infty \frac{1}{(nx)^{m/2}} \quad ,$$

(2.2.2) $$\leq M n^r \|f\|_\infty$$

Note that in the above use has been made of the constraints $2r-2l-m \geq 0$ and $m \geq 0$.

Since $x^r S_n^{(2r)} f(x)$ is a sum of terms $S(l,m,n,x)$

(2.2.3) $$|x^r S_n^{(2r)} f(x)| \leq M n^r \|f\|_\infty \quad , \quad x \geq 1/n$$

The inequalities (2.2.1) and (2.2.3) together give us the lemma . ∎

<u>Corollary 2.2.1:</u> For $f \in C[0,\infty)$ and $r \in \mathbb{N}$

$$\|S_n f\|_{d,2r} \leq M n^r \|f\|_\infty$$

<u>Proof:</u> Using (2.1.1) lemma 2.2.1 and the fact that

$$|S_n^{(r)} f(x)| \leq \sum_{k=0}^{\infty} |\Delta_{1/n}^r f(\tfrac{k}{n})| \, q_{n,k}(x) \leq 2^r n^r \|f\|_\infty ,$$

the corollary follows. ∎

2.3 BERNSTEIN TYPE INEQUALITY FOR $f \in \mathcal{D}_{2r}$

<u>LEMMA 2.3.1</u>: For $f \in \mathcal{D}_{2r}$

$$\|x^r S_n^{(2r)} f\|_\infty \leq M \|x^r f^{(2r)}\|_\infty$$

<u>Proof</u>: Since $f \in \mathcal{D}_{2r}$, using mean value theorem , for $k \geq 1$

$$n^{2r} |\Delta_{1/n}^{2r} f(\tfrac{k}{n})| \leq \sup_{\frac{k}{n} < \xi < \frac{k+2r}{n}} |f^{(2r)}(\xi)|$$

$$\leq \|\xi^r f^{(2r)}\|_\infty \sup_{\frac{k}{n} < \xi < \frac{k+2r}{n}} \frac{1}{\xi^r}$$

(2.3.1) $$\leq (\tfrac{n}{k})^r \|\xi^r f^{(2r)}\|_\infty$$

Further for $k = 0$, using Taylor's expansion with integral form of remainder

$$\Delta^{2r}_{1/n} f(0) = \sum_{j=0}^{2r} \binom{2r}{j} (-1)^j f(\tfrac{j}{n})$$

$$= \sum_{l=1}^{2r-1} \frac{(-1)^{l+1}}{l!} \sum_{j=1}^{2r} \binom{2r}{j} (-1)^j (\tfrac{j}{n})^l f^{(l)}(\tfrac{j}{n})$$

$$+ \frac{1}{(2r-1)!} \sum_{j=1}^{2r} \binom{2r}{j} (-1)^j \int_0^{\frac{j}{n}} u^{2r-1} f^{(2r)}(u)du$$

(2.3.2) $$= \sum_{l=1}^{2r-1} S_l + I \text{ , say.}$$

Now ,

$$|I| \le \frac{1}{(2r-1)!} \sum_{j=1}^{2r} \binom{2r}{j} \int_0^{j/n} u^{2r-1} |f^{(2r)}(u)|du$$

(2.3.3) $$\le M n^{-r} \|u^r f^{(2r)}\|_\infty$$

and using $j^l = \sum_{i=0}^{l-1} a_i \, j \, (j-1) \, . \, . \, . \, (j-i)$

$$S_l = \frac{(-1)^{l+1}}{l! \; n^l} \sum_{j=1}^{2r} \binom{2r}{j} (-1)^j \sum_{i=0}^{l-1} a_i \; j(j-1) \quad (j-i) f^{(l)}(\tfrac{j}{n})$$

$$= \frac{(-1)^{l+1}}{l! \; n^l} \sum_{i=0}^{l-1} a_i \; 2r \; ... \; (2r-i) \sum_{j=i+1}^{2r} \binom{2r-i-1}{j-i-1} (-1)^j f^{(l)}(\tfrac{j}{n})$$

$$= \frac{(-1)^{l+1}}{l!\, n^{l}} \sum_{i=0}^{l-1} a_i \; 2r \ldots (2r-i)$$

$$\sum_{j=0}^{2r-i-1} \binom{2r-i-1}{j} (-1)^{j+i+1} f^{(l)}\left(\frac{j+i+1}{n}\right)$$

$$= \frac{(-1)^{l+1}}{l!\, n^{l}} \sum_{i=0}^{l-1} a_i \; 2r \ldots (2r-i) \quad (-1)^{i+1} \Delta_{1/n}^{2r-i-1} f^{(l)}\left(\frac{i+1}{n}\right)$$

Thus ,

$$|S_l| \le M\, n^{-l} \sum_{i=0}^{l-1} \left|\Delta_{1/n}^{2r-i-1} f^{(l)}\left(\frac{i+1}{n}\right)\right|$$

$$(2.3.4) \qquad \le M\, n^{-l} \sum_{i=0}^{l-1} M \sup_{0\le j\le l-i-1} \left|\Delta_{1/n}^{2r-l} f^{(l)}\left(\frac{j+i+1}{n}\right)\right| .$$

Therefore, using mean value theorem,

$$n^{2r-l} \left|\Delta_{1/n}^{2r-l} f^{(l)}\left(\frac{j+i+1}{n}\right)\right|$$

$$\le \sup_{\frac{i+j+1}{n} < t < \frac{i+j+1+2r-l}{n}} |f^{(2r)}(t)|$$

$$\le n^{r} \| t^{r} f^{(2r)} \|_{\infty}$$

Hence ,

$$(2.3.5) \qquad |S_l| \le M\, n^{-r} \| t^{r} f^{(2r)} \|_{\infty}$$

Inequalities (2.3.3) and (2.3.5) together imply

$$(2.3.6) \qquad n^{2r} |\Delta_{1/n}^{2r} f(0)| \le M\, n^{r} \| x^{r} f^{(2r)} \|_{\infty} .$$

So ,

$$x^{r} |S_n^{(2r)} f(x)| \le n^{2r}\, x^{r} \sum_{k=0}^{\infty} \left|\Delta_{1/n}^{2r} f\left(\frac{k}{n}\right)\right| q_{n,k}(x)$$

$$= x^r \left[n^{2r} |\Delta_{1/n}^{2r} f(0)| q_{n,0}(x) \right.$$

$$\left. + n^{2r} \sum_{k=1}^{\infty} |\Delta_{1/n}^{2r} f(\tfrac{k}{n})| \; q_{n,k}(x) \right]$$

$$\leq M \; x^r \; \|x^r f^{(2r)}\|_\infty \left[n^r q_{n,0}(x) + \sum_{k=1}^{\infty} q_{n,k}(x) \; (\tfrac{n}{k})^r \right]$$

(by (2.3.1) and (2.3.6))

$$\leq M \; x^r \; \|x^r f^{(2r)}\|_\infty \sum_{k=0}^{\infty} q_{n,k}(x) \; (\tfrac{n}{k+1})^r \; .$$

From here the result follows by lemma 2.1.2. ∎

Corollary 2.3.1: For $f \in \mathcal{D}_{2r}$

$$\|S_n f\|_{d,2r} \leq M \; \|f\|_{d,2r}$$

Proof: By mean value theorem

$$|S_n^{(r)} f(x)| \;\leq\; n^r \sum_{k=0}^{\infty} |\Delta_{1/n}^{r} f(\tfrac{k}{n})| \; q_{n,k}(x) \;\leq\; \|f^{(r)}\|_\infty$$

This together with (2.1.1) and lemma 2.3.1 gives the corollary. ∎

2.4 DIRECT THEOREM FOR $f \in \mathcal{D}_{2r}$

THEOREM 2.4.1: For $f \in \mathcal{D}_{2r}$

$$\|S_{n,r} f - f\|_\infty \leq M \; n^{-r} \; \|f\|_{d,2r}$$

Proof: We will prove the theorem in two parts.

Case a: Let $x \geq A/n$, A is some constant independent of n .

Using Taylor's expansion, with integral form of remainder

$$S_{n,r} f(x) - f(x) = \sum_{i=0}^{r-1} C(i,r) \sum_{k=0}^{\infty} q_{n_i,k}(x) \left[f(\tfrac{k}{n_i}) - f(x) \right]$$

$$= \sum_{l=1}^{2r-1} \frac{1}{l!} f^{(l)}(x) \sum_{i=0}^{r-1} C(i,r) \sum_{k=0}^{\infty} q_{n_i,k}(x) \left[\frac{k}{n_i} - x\right]^{l}$$

$$+ \frac{1}{(2r-1)!} \sum_{i=0}^{r-1} C(i,r) \sum_{k=0}^{\infty} q_{n_i,k}(x) \int_x^{\frac{k}{n_i}} \left[\frac{k}{n_i} - u\right]^{2r-1} f^{(2r)}(u)du.$$

(2.4.1) $\quad = S_1(x) + S_2(x)$, say

Therefore, by lemma 2.1.4 ,

$$S_1(x) = \sum_{l=2}^{2r-1} \frac{1}{l!} f^{(l)}(x) \sum_{i=0}^{r-1} C(i,r)\, n_i^{-l}\, T_{n_i,l}(x)$$

$$= \sum_{l=2}^{2r-1} \frac{1}{l!} f^{(l)}(x) \sum_{i=0}^{r-1} C(i,r)\, n_i^{-l} \sum_{j=1}^{[l/2]} a_{j,l}\, (n_i x)^{j}$$

$$= \sum_{l=2}^{r} \frac{1}{l!} f^{(l)}(x) \sum_{j=1}^{[l/2]} a_{j,l}\, x^{j} \sum_{i=0}^{r-1} C(i,r)\, n_i^{j-l}$$

$$+ \sum_{l=r+1}^{2r-1} \frac{1}{l!} f^{(l)}(x) \sum_{j=1}^{[l/2]} a_{j,l}\, x^{j} \sum_{i=0}^{r-1} C(i,r)\, n_i^{j-l}$$

Now for $2 \le l \le r$ we have $1 \le l-j \le r-1$, for all possible values of j , thus, using (1.5.1d) and the fact that $T_{n,1} \equiv 0$, we have

<u>Remark</u> <u>2.4.1</u>: For $1 \le l \le r$ is the corresponding part of $S_1(x)$ is identically equal to zero.

Thus effectively,

$$S_1(x) = \sum_{l=r+1}^{2r-1} \frac{1}{l!} f^{(l)}(x) \sum_{j=1}^{[l/2]} a_{j,l}\, x^{j} \sum_{i=0}^{r-1} C(i,r)\, n_i^{j-l}$$

Once again due to (1.5.1d) only those terms contribute to $S_1(x)$ for which $l-j \geq r$. So

$$S_1(x) = \sum_1 \frac{1}{l!} f^{(l)}(x)\, a_{j,l}\, x^j \sum_{i=0}^{r-1} C(i,r)\, n_i^{j-l}$$

where Σ_1 represents a sum over $r+1 \leq l \leq 2r-1$, $1 \leq j \leq [l/2]$ such that $l-j \geq r$. Now , (1.5.1a-b) together with the fact that $a_{j,l}$ are constants independent of n gives

$$|S_1(x)| \leq M \sum_1 x^j\, |f^{(l)}(x)|\, n^{j-l}$$

$$\leq M \sum_1 x^j\, |f^{(l)}(x)|\, n^{j-l} \left(\frac{nx}{A}\right)^{l-j-r}$$

(since $x \geq A/n$)

$$\leq M\, n^{-r} \sum_1 x^{l-r}\, |f^{(l)}(x)|$$

Now since the sum is over all those values of l and j such that $l-j \geq r, r+1 \leq l \leq 2r-1$ and $1 \leq j \leq [l/2]$, we have $1 \leq l-r \leq r-1$. Therefore, by remark 2.4.1 and corollary 2.1.2,

$$|S_1(x)| \leq M\, n^{-r}\left(M\, \|f^{(r)}\|_\infty + \|x^r f^{(2r)}\|_\infty \right)$$

$$\text{(2.4.2)} \qquad \leq M\, n^{-r}\, \|f\|_{d,2r}$$

Next we estimate,

$$S_2(x) = \frac{1}{(2r-1)!}\sum_{i=0}^{r-1} C(i,r) \sum_{k=0}^{\infty} q_{n_i,k}(x) \int_x^{\frac{k}{n_i}} \left[\frac{k}{n_i} - u\right]^{2r-1} f^{(2r)}(u)\,du$$

We consider the term corresponding to k = 0 separately.

$$S_{2,0}(x) = \frac{1}{(2r-1)!} \sum_{i=0}^{r-1} C(i,r)\, q_{n_i,0}(x) \int_x^0 (0-u)^{2r-1} f^{(2r)}(u)\,du$$

Since C(i,r) are constants independent of n

$$|S_{2,0}(x)| \le M \sum_{i=0}^{r-1} q_{n_i,0}(x) \int_0^x u^{2r-1} \,|f^{(2r)}(u)|\, du$$

$$\le M \,\|u^r f^{(2r)}\|_\infty \sum_{i=0}^{r-1} q_{n_i,0}(x)\, x^r$$

(2.4.3)
$$= M\, x^{-r} \|u^r f^{(2r)}\|_\infty \sum_{i=0}^{r-1} n_i^{-2r} (n_i x)^{2r}\, q_{n_i,0}(x)$$

The rest of the terms are treated together writing

$$S_{2,r}(x) = \frac{1}{(2r-1)!} \sum_{i=0}^{r-1} C(i,r) \sum_{k=1}^{\infty} q_{n_i,k}(x) \int_x^{\frac{k}{n_i}} \left(\frac{k}{n_i} - u\right)^{2r-1} f^{(2r)}(u)\,du$$

By (1.5.1b) ,

$$|S_{2,r}(x)| \le M \sum_{i=0}^{r-1} \sum_{k=1}^{\infty} q_{n_i,k}(x) \left|\frac{k}{n_i} - x\right|^{2r-1} \left|\int_x^{\frac{k}{n_i}} u^r |f^{(2r)}(u)|\, u^{-r} du\right|$$

$$\le M \,\|u^r f^{(2r)}\|_\infty \sum_{i=0}^{r-1} \sum_{k=1}^{\infty} q_{n_i,k}(x) \left|\frac{k}{n_i} - x\right|^{2r-1} \left|\int_x^{\frac{k}{n_i}} u^{-r} du\right|$$

$$\le M \,\|u^r f^{(2r)}\|_\infty \sum_{i=0}^{r-1} \sum_{k=1}^{\infty} q_{n_i,k}(x) \left|\frac{k}{n_i} - x\right|^{2r} \left[x^{-r} + \left(\frac{k}{n_i}\right)^{-r}\right]$$

(2.4.4)
$$= M \,\|u^r f^{(2r)}\|_\infty\, x^{-r} \sum_{i=0}^{r-1} n_i^{-2r} \sum_{k=1}^{\infty} (k - n_i x)^{2r}\, q_{n_i,k}(x)$$

$$+ M \,\|u^r f^{(2r)}\|_\infty \sum_{i=0}^{r-1} n_i^{-2r} \sum_{k=1}^{\infty} (k - n_i x)^{2r} q_{n_i,k}(x) \left(n_i/k\right)^r$$

From (2.4.3-4),

$$|S_2(x)| \le M \|u^r f^{(2r)}\|_\infty x^{-r} \sum_{i=0}^{r-1} n_i^{-2r} \sum_{k=0}^{\infty} (k-n_i x)^{2r} q_{n_i,k}(x)$$

$$+ M \|u^r f^{(2r)}\|_\infty \sum_{i=0}^{r-1} n_i^{-2r} \sum_{k=1}^{\infty} (k-n_i x)^{2r} q_{n_i,k}(x) (n_i/k)^r$$

(2.4.5) $\quad = I_1(x) + I_2(x)$, say

By Cauchy - Schwarz inequality,

$$|I_2(x)| \le M \|u^r f^{(2r)}\|_\infty \sum_{i=0}^{r-1} n_i^{-2r} \left[\sum_{k=0}^{\infty} (k-n_i x)^{4r} q_{n_i,k}(x) \right]^{1/2}$$

$$\cdot \left[\sum_{k=0}^{\infty} \left(\frac{n_i}{(k+1)}\right)^{2r} q_{n_i,k}(x) \right]^{1/2}$$

Applying lemmas 2.1.2 and 2.1.4 the right hand side can be shown not to exceed

$$M \|u^r f^{(2r)}\|_\infty x^{-r} \sum_{i=0}^{r-1} n_i^{-2r} (T_{n_i,4r}(x))^{1/2}$$

Since $x \ge \frac{A}{n}$, by corollary 2.1.1 and (1.5.1a),

(2.4.6) $\quad |I_2(x)| \le M n^{-r} \|u^r f^{(2r)}\|_\infty$

In a similar fashion it can be shown that

(2.4.7) $\quad |I_1(x)| \le M n^{-r} \|u^r f^{(2r)}\|_\infty$

Now,

(2.4.8) $\quad |S_2(x)| \le M n^{-r} \|f\|_{d,2r}$,

follows from (2.4.6-7).

Further, inequalities (2.4.8) and (2.4.2) along with (2.4.1) give

$$|S_{n,r}f(x) - f(x)| \le M \quad n^{-r} \; \|f\|_{d,2r} \quad \text{for } x \ge A/n \,.$$

Thus completing case a .

<u>Case</u> <u>b</u>: Let $x \le A/n$

Here again by Taylor's expansion with integral form of remainder

$$S_{n,r}f(x)-f(x) = \sum_{l=1}^{r-1} \frac{1}{l!} f^{(l)}(x) \sum_{i=0}^{r-1} C(i,r) \sum_{k=0}^{\infty} q_{n_i,k}(x)\left(\frac{k}{n_i} - x\right)^l$$

$$+ \frac{1}{(r-1)!} \sum_{i=0}^{r-1} C(i,r) \sum_{k=0}^{\infty} q_{n_i,k}(x) \int_x^{\frac{k}{n_i}} \left(\frac{k}{n_i} - u\right)^{r-1} f^{(r)}(u)du$$

$$= S_1(x) + S_2(x), \text{ say.}$$

That $S_1(x) = 0$ is a consequence of remark 2.4.1. Hence all we need is to estimate $S_2(x)$. Using (1.5.1b),

$$|S_2(x)| \le M \sum_{i=0}^{r-1} \sum_{k=0}^{\infty} q_{n_i,k}(x) \left|\frac{k}{n_i} - x\right|^{r-1} \left| \int_x^{\frac{k}{n_i}} |f^{(r)}(u)| \, du\right|$$

$$\le M \; \|f^{(r)}\|_\infty \sum_{i=0}^{r-1} \sum_{k=0}^{\infty} q_{n_i,k}(x) \left|\frac{k}{n_i} - x\right|^r$$

$$\le M \; \|f\|_{d,2r} \sum_{i=0}^{r-1} n_i^{-r} \sum_{k=0}^{\infty} q_{n_i,k}(x) |k - n_i x|^r$$

Applying Cauchy-Schwarz inequality to the right hand side we get

$$|S_2(x)| \le M \; \|f\|_{d,2r} \sum_{i=0}^{r-1} n_i^{-r} \left[\sum_{k=0}^{\infty} q_{n_i,k}(x) \; (k - n_i x)^{2r}\right]^{1/2}$$

An application of lemma 2.1.4 gives

$$|S_2(x)| \le M \; \|f\|_{d,2r} \sum_{i=0}^{r-1} n_i^{-r} \left[\sum_{j=1}^{r} |a_{j,2r}| (n_i x)^j \right]^{1/2}$$

Now, (1.5.1a) together with the fact that $x \le A/n$ implies

$$|S_2(x)| \le M\, n^{-r}\, \|f\|_{d,2r}$$

This completes the estimate for case b.

Hence the theorem.∎

2.5 INVERSE THEOREM

<u>THEOREM</u> <u>2.5.1</u>: For $f \in C[0,\infty)$ and $\phi \in \Phi_{2r}$

$$\|S_{n,r}f - f\|_\infty = O(\phi(n^{-1/2})) \text{ if and only if } f \in \mathcal{D}_{2r,\phi}$$

<u>Proof</u>: The symbols C_1, C_2 etc. appearing in the proof are taken to be constants independent of n and t.

From corollaries 2.2.1 and 2.3.1,

$$\text{(2.5.1a)} \qquad \|S_n f\|_{d,2r} \le M\, n^r\, \|f\|_\infty, \qquad f \in C[0,\infty)$$

$$\text{(2.5.1b)} \qquad \|S_n f\|_{d,2r} \le M\, \|f\|_{d,2r}, \qquad f \in \mathcal{D}_{2r}$$

Since

$$\|S_{n,r}f\|_{d,2r} = \Big\| \sum_{i=0}^{r-1} C(i,r) S_{n_i} f \Big\|_{d,2r}$$

$$\le \sum_{i=0}^{r-1} |C(i,r)|\, \|S_{n_i} f\|_{d,2r}$$

$$\text{(2.5.1c)} \qquad \le M \max_{0 \le i \le r} \| S_{n_i} f \|_{d,2r}$$

Inequalities (2.5.1a-b) give

$$\text{(2.5.2a)} \qquad \|S_{n,r}f\|_{d,2r} \le M\, n^r\, \|f\|_\infty, \qquad f \in C[0,\infty)$$

$$\text{(2.5.2b)} \qquad \|S_{n,r}f\|_{d,2r} \le M\, \|f\|_{d,2r}, \qquad f \in \mathcal{D}_{2r}$$

Suppose $\|S_{n,r}f - f\|_\infty = O(\phi(n^{-1/2}))$. Then

$$K(t^{2r}, f) = \inf_{g \in D_{2r}} \left\{ \|f-g\|_\infty + t^{2r} \|g\|_{d,2r} \right\}$$

$$\leq \left\{ \|f-S_{n,r}f\|_\infty + t^{2r}\|S_{n,r}f\|_{d,2r} \right\}$$

$$\leq \left\{ C_1\ \phi(n^{-1/2}) + t^{2r}\left\{\|S_{n,r}(f-g)\|_\infty + t^{2r}\|S_{n,r}g\|_{d,2r} \right\} \right\} \quad (g \in \mathcal{D}_{2r})$$

Therefore, by (2.5.2a-b),

$$K(t^{2r},f) \leq C_2 \left\{ \phi(n^{-1/2}) + t^{2r}n^{-r}\left\{ \| f-g \|_\infty + \|g\|_{d,2r} \right\} \right\}$$

$$\leq C_3 \left\{ \phi(n^{-1/2}) + t^{2r}n^{-r}K(n^{-r},f) \right\}$$

Thus, by lemma 1.6.3,

$$K(t^{2r},f) = O(\phi(t))$$

which is same as $f \in \mathcal{D}_{2r,\phi}$.

Conversely suppose $f \in \mathcal{D}_{2r,\phi}$, for any t there is a $g_t \in \mathcal{D}_{2r}$ such that

(2.5.3a) $\|f-g_t\|_\infty \leq C_4.\phi(t)$, and

(2.5.3b) $t^{2r}\|g_t\|_{d,2r} \leq C_4\ \phi(t)$

Now,

$$\|S_{n,r}f - f\|_\infty \leq \|S_{n,r}(f-g_t)\|_\infty + \|S_{n,r}g_t-g_t\|_\infty + \|g_t-f\|_\infty$$

$$\leq (M+1)\ \|f-g_t\|_\infty + \|S_{n,r}g_t-g_t\|_\infty$$

$$\leq C_4(M+1)\phi(t) + \|S_{n,r}g_t-g_t\|_\infty$$

Since $g_t \in \mathcal{D}_{2r}$, by theorem 2.4.1 and (2.5.3b),

$$\|S_{n,r}f - f\|_\infty \leq C_4(M+1)\phi(t) + M\ C_4\ n^{-r}\ t^{-2r}\ \phi(t)$$

Taking $t = n^{-1/2}$

$$\|S_{n,r}f-f\|_\infty = O(\phi(n^{-1/2}))$$

Hence the theorem . ∎

CHAPTER III

GLOBAL SIMULTANEOUS APPROXIMATION BY LINEAR COMBINATIONS OF SZÁSZ-MIRAKJAN-HILLE OPERATOR

3.1 PRELIMINARIES

Simultaneous approximation by linear combinations of Szász-Mirakjan-Hille operator $S_n f(x)$ for $f \in C^m[0,\infty)$ is the object of study in this chapter.

An application of mean value theorem to lemma 2.1.3 gives

$$(3.1.1) \qquad S_n^{(m)} f(x) = \sum_{k=0}^{\infty} q_{n,k}(x)\ f^{(m)}\left(\frac{k+m\theta_k}{n}\right), \qquad 0 < \theta_k < 1.$$

It can be easily verified that

$$(3.1.2) \qquad \|S_n^{(m)} f\|_\infty \le \|f^{(m)}\|_\infty$$

where $\|f\|_\infty = \sup_{x\ge 0} |f(x)|$.

Following two lemmas are generalizations of the lemmas 2.2.1 and 2.3.1 respectively .

<u>LEMMA</u> <u>3.1.1</u>: Let $f \in C^m[0,\infty)$ and $r \in \mathbb{N}$, then

$$\|x^r S_n^{(m+2r)} f\|_\infty \le M\ n^r \|f^{(m)}\|_\infty$$

<u>Proof</u>: We have from lemma 2.1.3 and using mean value theorem that

$$S_n^{(m+2r)} f(x) = n^{m+2r} \sum_{k=0}^{\infty} q_{n,k}(x)\ \Delta_{\frac{1}{n}}^{m+2r} f\left(\frac{k}{n}\right)$$

$$= n^{2r} \sum_{k=0}^{\infty} q_{n,k}(x) \Delta_{\frac{1}{n}}^{2r} f^{(m)}\left(\frac{k+m\theta_k}{n}\right) \quad , \qquad 0 < \theta_k < 1 .$$

Following the same steps as in lemma 2.2.1 for $x \le 1/n$,

(3.1.3a) $\quad |x^r S_n^{(m+2r)} f(x)| \le M\, n^r \|f^{(m)}\|_\infty \qquad x \le 1/n$.

Also , since

$$x^r S_n^{(m+2r)} f(x) = x^r \sum_{k=0}^{\infty} q_{n,k}^{(2r)}(x) f^{(m)}\left(\frac{k+m\theta_k}{n}\right) \quad , \qquad 0 < \theta_k < 1$$

using lemma 2.1.1 , $x^r S_n^{(m+2r)} f(x)$ can be expressed as a sum of terms of the type

$$c(2r,l,s)\; x^r \frac{n^l}{x^{2r-l}} \sum_{k=0}^{\infty} (k-nx)^{2r-2l-s} q_{n,k}(x) f^{(m)}\left(\frac{k+m\theta_k}{n}\right) .$$

Again proceeding in the same fashion, as in lemma 2.2.1 for $x \ge 1/n$,

(3.1.3b) $\quad |x^r S_n^{(m+2r)} f(x)| \le M\, n^r \|f^{(m)}\|_\infty \qquad x \ge 1/n$.

Inequalities (3.1.3a-b) together give us the lemma . ∎

<u>LEMMA</u> <u>3.1.2</u>: Let $f^{(m)} \in \mathcal{D}_{2r}$. Then

$$\|x^r S_n^{(m+2r)} f\|_\infty \le M\, \|x^r f^{(m+2r)}\|_\infty$$

<u>Proof</u>:

$$|x^r S_n^{(m+2r)} f(x)| \le x^r n^{m+2r} \sum_{k=0}^{\infty} q_{n,k}(x)\, |\Delta_{\frac{1}{n}}^{m+2r} f(\tfrac{k}{n})|$$

For $k \ge 1$, by mean value theorem,

$$n^{m+2r} |\Delta_{\frac{1}{n}}^{m+2r} f(\tfrac{k}{n})| \le \sup_{\frac{k}{n} < \xi < \frac{k+m+2r}{n}} |f^{(m+2r)}(\xi)|$$

$$\le \|\xi^r f^{(m+2r)}\|_\infty \sup_{\frac{k}{n} < \xi < \frac{k+m+2r}{n}} \frac{1}{\xi^r}$$

$$\le \left(\frac{n}{k}\right)^r \|\xi^r f^{(m+2r)}\|_\infty . \qquad (3.1.4)$$

For k = 0 following the same line as in lemma 2.3.1 for k=0

$$n^{m+2r} |\Delta_{\frac{1}{n}}^{m+2r} f(0)| \le n^r \|x^r f^{(m+2r)}\|_\infty .$$

With this and (3.1.4) , it can be shown, as in lemma 2.3.1 ,that

$$|x^r S_n^{(m+2r)} f| \le M \|x^r f^{(m+2r)}\|_\infty \qquad x \in [0,\infty)$$

and hence the lemma.■

3.2 DIRECT THEOREM

Before proving the main theorem of this section, which is a direct theorem for simultaneous approximation by linear combination of $S_n f$, $f^{(m)} \in \mathcal{D}_{2r}$, we will prove some preliminary results which are used to establish the above mentioned theorem.

LEMMA 3.2.1: Let $m \in \mathbb{N}$ and $l \in \mathbb{N}_o$. Then for any $a,h \in \mathbb{R}$

$$\int_a^{a+h} (a+h-t)^{m-1}(t-x)^l dt = \sum_{\nu=0}^{l} \frac{(m-1)!}{(m+\nu)!} \frac{l!}{(l-\nu)!} h^{m+\nu} (a-x)^{l-\nu}$$

Proof: Repeated use of integration by parts gives the lemma.■

LEMMA 3.2.2: Let $r,n \in \mathbb{N}$ and $x \in [0,\infty)$ then

$$e^{-nx} x^r \le \frac{r!}{n^r}$$

Proof: Since $e^{-nx} \frac{(nx)^r}{r!} \le 1$ the lemma follows .■

LEMMA 3.2.3: Let $f \in C^m[0,\infty)$. Then

$$\Delta_h^m f(x) = \frac{1}{(m-1)!} \sum_{s=0}^{m} \binom{m}{s} (-1)^{m-s} \int_x^{x+h} (x+h-t)^{m-1} f^{(m)}(t)dt$$

where $\Delta f(x) = f(x+h) - f(x)$.

Proof: Since

$$(3.2.1)\quad \sum_{s=0}^{m} \binom{m}{s} (-1)^{m-s} s^k = \begin{cases} 0 & 0 \le k < m \\ m! & k=m \end{cases}$$

the lemma follows by Taylors expansion . ∎

Now we take up the main theorem of this section .

THEOREM 3.2.1: Let $m \in \mathbb{N}$ and $f^{(m)} \in \mathcal{D}_{2r}$, then

$\|S_{n,r}^{(m)} f - f^{(m)}\|_\infty \le M\, n^{-r} \|f^{(m)}\|_{d,2r}$.

Proof: By lemmas 2.1.3 and 3.2.3,

$$S_{n,r}^{(m)} f(x) - f^{(m)}(x) = \sum_{i=0}^{r-1} C(i,r)\, (S_{n_i}^{(m)} f(x) - f^{(m)}(x))$$

$$= \sum_{i=0}^{r-1} C(i,r) \Big[n_i^m \sum_{k=0}^{\infty} q_{n_i,k}(x) . \Delta^m_{\frac{1}{n_i}} f(\tfrac{k}{n_i}) - f^{(m)}(x) \Big]$$

$$= \sum_{i=0}^{r-1} C(i,r)\, n_i^m \frac{1}{(m-1)!} \sum_{s=0}^{m} \binom{m}{s} (-1)^{m-s} \sum_{k=0}^{\infty} q_{n_i,k}(x)$$

$$. \int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} (\tfrac{k+s}{n_i} - t)^{m-1} (f^{(m)}(t) - f^{(m)}(x))dt$$

$$+ \sum_{i=0}^{r-1} \Big[n_i^m \frac{1}{(m-1)!} \sum_{s=0}^{m} \binom{m}{s} (-1)^{m-s} \sum_{k=0}^{\infty} q_{n_i,k}(x)$$

$$\int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} (\tfrac{k+s}{n_i} - t)^{m-1} dt - 1 \Big] f^{(m)}(x)$$

$$(3.2.2)\quad = S_1(x) + S_2(x), \text{ say.}$$

Remark 3.2.1: That $S_2(x) = 0$ follows immediately using lemma 3.2.1 and (3.2.1) along with the fact that $\sum_{k=0}^{\infty} q_{n,k}(x) = 1$.

So we need only concentrate on $S_1(x)$. We consider two cases.

Case i: Let $x \geq A/n$, A some constant. Using Taylor's expansion with integral form of remainder

$$S_1(x) = \frac{1}{(m-1)!} \sum_{s=0}^{m} \binom{m}{s} \sum_{i=0}^{r-1} C(i,r)\, n_i^m \sum_{k=0}^{\infty} q_{n_i,k}(x)$$

$$\cdot \sum_{l=1}^{2r-1} \frac{1}{l!} f^{(m+l)}(x) . \int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} \left(\frac{k+s}{n_i} - t\right)^{m-1} (t - x)^l dt$$

$$+ \frac{1}{(m-1)!} \frac{1}{(2r-1)!} \sum_{s=0}^{m} \binom{m}{s} (-1)^{m-s} \sum_{i=0}^{r-1} C(i,r) n_i^m \sum_{k=0}^{\infty} q_{n_i,k}(x)$$

$$\cdot \int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} \left(\frac{k+s}{n_i} - t\right)^{m-1} \int_x^t (t - u)^{2r-1} . f^{(m+2r)}(u) du dt.$$

$$(3.2.3) \qquad = S_{11}(x) + S_{12}(x).$$

For $s = 0$ the limits of the integration over t become equal, consequently, the term corresponding to $s = 0$ is identically equal to zero. So here onwards we will have $s \geq 1$.

Lemma 3.2.1 with $a = k/n_i$ and $h = s/n_i$ gives

$$S_{11}(x) = \frac{1}{(m-1)!} \sum_{s=1}^{m} \sum_{l=1}^{2r-1} \binom{m}{s} (-1)^{m-s} \frac{1}{l!} f^{(m+l)}(x) \sum_{\nu=0}^{l} \frac{(m-1)! \quad l!}{(m+\nu)!(l-\nu)!} s^{m+\nu}$$

$$\sum_{i=0}^{r-1} C(i,r)\, n_i^m \frac{1}{n_i^{l+m}} \sum_{k=0}^{\infty} q_{n_i,k}(x)\, (k-n_i x)^{l-\nu}$$

$$(3.2.4) = \sum_{s=1}^{m} \sum_{l=1}^{2r-1} \sum_{\nu=0}^{l} A(m,s,l,\nu) f^{(m+l)}(x) \sum_{i=0}^{r-1} C(i,r) \frac{1}{n_i^l} T_{n_i,l-\nu}(x)$$

where $T_{n_i,l-\nu}(x)$ is as in lemma 2.1.4 and

$$A(m,s,l,\nu) = \binom{m}{s} \frac{1}{(m+\nu)!} \frac{1}{(l-\nu)!} (-1)^{m-s} s^{m+\nu}$$

let

$$S(l,\nu,x) = f^{(m+l)}(x) \sum_{i=0}^{r-1} C(i,r) \frac{1}{n_i^l} T_{n_i,l-\nu}(x)$$

It will be shown that

$$(3.2.5) \quad |S(l,\nu,x)| \le M \frac{1}{n^r} \|f^{(m)}\|_{d,2r}, \quad 1 \le l \le 2r-1 \text{ and } 0 \le \nu \le l .$$

This together with (3.2.4) and the fact that $A(m,s,l,\nu)$ is bounded by a constant independent of n gives

$$(3.2.6) \quad |S_{11}(x)| \le M \frac{1}{n^r} \|f^{(m)}\|_{d,2r}$$

So all we need is to establish (3.2.5).

Invoking part (i) of lemma 2.1.4,

$$(3.2.7a) \quad S(l,l-1,x) = 0 \text{ and}$$

$$S(l,l,x) = f^{(m+l)}(x) \sum_{i=0}^{r-1} C(i,r) \frac{1}{n_i^l}$$

Thus , due to (1.5.1d),

$$(3.2.7b) \quad S(l,l,x) = 0 \quad , \qquad 1 \le l \le r-1 .$$

For, $r \le l \le 2r-1$,

$$|S(l,l,x)| \le |f^{(m+l)}(x)| \sum_{i=0}^{r-1} |C(i,r)| \frac{1}{n_i^l}$$

$$\le M \frac{1}{n^l} |f^{(m+l)}(x)|$$

$$(3.2.8) \qquad \le M \frac{1}{n^{l}} \left(\frac{nx}{A}\right)^{l-r} |f^{(m+l)}(x)| \qquad (\text{ since } x \ge A/n)$$

Now when $l = r$,

$$|f^{(m+l)}(x)| = |f^{(m+r)}(x)| \le \|f^{(m+r)}\|_{\infty}$$

and when $r < l \le 2r-1$, by lemma 2.1.5,

$$|x^{l-r} f^{(m+l)}(x)| \le M \|f^{(m+r)}\|_{\infty} + \|x^{r} f^{(m+2r)}\|_{\infty}$$

Consequently , from (3.2.8),

$$(3.2.9) \qquad |S(l,l,x)| \le M \frac{1}{n^{r}} \|f^{(m)}\|_{d,2r} \quad , \quad r \le l \le 2r-1 \ .$$

So far we have taken care of the cases when $\nu = l-1, l$ and $1 \le l \le 2r-1$. Note that for $l = 1$ only values of ν are l and $l-1$, so, we need not consider the case $l = 1$ now onwards. We have yet to estimate $S(l,\nu,x)$ for $2 \le l \le 2r-1$ and $0 \le \nu \le l-2$. But the range of ν allows us to write

$$S(l,\nu,x) = f^{(m+l)}(x) \sum_{i=0}^{r-1} C(i,r) \frac{1}{n_i^{l}} \sum_{j=1}^{[\frac{l-\nu}{2}]} a_{j,l-\nu} (n_i x)^{j}$$

Lemma 2.1.4 has been used to get the above expression . For $2 \le l \le r$, we have that $1 \le l-j \le r-1$ and so using (1.5.1d),

$$(3.2.10) \quad S(l,\nu,x) = 0 \quad , \quad 2 \le l \le r \ .$$

Let $r+1 \le l \le 2r-1$. Again due to (1.5.1d) only those terms in $S(l,\nu,x)$ will be non zero for which $l-j \ge r$. So let $l-j \ge r$, $r+1 \le l \le 2r-1$, $0 \le \nu \le l-2$. Thus ,

$$|S(l,\nu,x)| = \sum_{j} |a_{j,l-\nu}| x^{j} |f^{(m+l)}(x)| \sum_{i=0}^{r-1} |C(i,r)| \frac{1}{n_i^{l-j}}$$

$$\le M \sum_{j} x^{j} |f^{(m+l)}(x)| \frac{1}{n^{l-j}}$$

where the sum is over all those j's for which $l-j \ge r$, $1 \le j \le [\frac{l-\nu}{2}]$, $r+1 \le l \le r-1$ and $0 \le \nu \le l-2$. Let $l-j = r+k$, $k \in \mathbb{N}_o$. Using that $x \ge A/n$, we have

$$\frac{x^j}{n^{l-j}}|f^{(m+l)}(x)| \le \frac{x^j}{n^{l-j}}|f^{(m+l)}(x)|(\frac{nx}{A})^k \le M\frac{x^{j+k}}{n^r}|f^{(m+r+k+j)}(x)|$$

$$(3.2.11) \qquad \le M\frac{1}{n^r}(M\|f^{(m+r)}\|_\infty + \|x^r f^{(m+2r)}\|_\infty)$$

This happens because $l = r+j+k$ and $r+1 \le l \le 2r-1$ implies that $1 \le j+k \le r-1$ and so lemma 2.1.5 can be applied to get the above inequality.

In view of (3.2.10-11),

$$|S(l,\nu,x)| \le M\frac{1}{n^r}\|f^{(m)}\|_{d,2r}, \quad 0 \le \nu \le l-2, \ 2 \le l \le 2r-1$$

This together with (3.2.7 a-b) and (3.2.9) establishes (3.2.5) and hence (3.2.6).

<u>Remark 3.2.2</u>: Inequalities (3.2.7 a-b) and (3.2.10) imply that $S(l,\nu,x) = 0$ for all $x \in [0,\infty)$ where $1 \le l \le r-1$.

We have yet to estimate $S_{12}(x)$ which we now take up.

Recall that

$$S_{12}(x) = \frac{1}{(m-1)!}\frac{1}{(2r-1)!}\sum_{s=0}^{m}\left[\begin{matrix} m \\ s\end{matrix}\right](-1)^{m-s}\sum_{i=0}^{r-1}C(i,r)\,n_i^m$$

$$\sum_{k=0}^{\infty} q_{n_i,k}(x).\int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}}(\frac{k+s}{n_i}-t)^{m-1}\int_x^t(t-u)^{2r-1}.f^{(m+2r)}(u)dudt.$$

Thus,

$$|S_{12}(x)| \leq M \sum_{s=0}^{m} \sum_{i=0}^{r-1} n_i \sum_{k=0}^{\infty} q_{n_i,k}(x) .$$

$$\cdot \int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} \left| \int_x^t |t - u|^{2r-1} . |f^{(m+2r)}(u)| du \right| dt .$$

$$(3.2.12) = M \sum_{s=0}^{m} \sum_{i=0}^{r-1} S(s,i,x) \text{, say.}$$

It is sufficient to estimate

$$S(s,i,x) = n_i \sum_{k=0}^{\infty} q_{n_i,k}(x) \int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} \left| \int_x^t |t-u|^{2r-1} |f^{(m+2r)}(u)| du \right| dt$$

where $1 \leq s \leq m$, $0 \leq i \leq r-1$. Since the term corresponding to $s = 0$ is identically zero , we do not bother about it. We consider the term corresponding to $k = 0$ separately. For $k = 0$ the corresponding term is

$$(3.2.13) \quad S_o(s,i,x) = n_i \; e^{-n_i x} \int_0^{\frac{s}{n_i}} \left| \int_x^t |t- u|^{2r-1} |f^{(m+2r)}(u)| du \right| dt$$

Since $x \geq A/n \geq A/n_i$ $(0 \leq i \leq r-1)$ and A is some constant. Choosing $A = 1$, we have $A/n_i \leq s/n_i$, $1 \leq s \leq m$. Therefore, splitting the integral over $0 \leq t \leq s/n_i$, in two parts $0 \leq t \leq \frac{A}{n_i}$ and $A/n_i \leq t \leq s/n_i$, we have $0 \leq t \leq u \leq x$, when $0 \leq t \leq \frac{A}{n_i}$. Consequently, $|t-u| \leq u \leq x$. Hence for $0 \leq t \leq A/n_i$ the

corresponding part of (3.2.13) is less than

$$n_i e^{-n_i x} \int_0^{\frac{A}{n_i}} \int_t^x |t-u|^{r-1} u^r \, |f^{(m+2r)}(u)| dudt$$

$$\leq n_i e^{-n_i x} \|u^r f^{(m+2r)}\|_\infty \int_0^{\frac{A}{n_i}} (x-t)^r dt$$

$$\leq A e^{-n_i x} x^r \|f^{(m)}\|_{d,2r}$$

$$\leq M \frac{1}{n_i^r} \|f^{(m)}\|_{d,2r} \qquad \text{(by lemma 3.2.2).}$$

The other part of (3.2.13) is when $A/n_i \leq t \leq s/n_i$. Here if $x \geq s/n_i$ then on the same lines as above we have

$$n_i e^{-n_i x} \int_{\frac{A}{n_i}}^{\frac{s}{n_i}} \left| \int_x^t |t-u|^{2r-1} \, |f^{(m+2r)}(u)| du \right| dt \leq M \frac{1}{n_i^r} \|f^{(m)}\|_{d,2r}$$

what still remains to be analyzed , in order to complete the estimate of (3.2.13) is the case when $A/n_i \leq t, x \leq s/n_i$. Here

$$n_i e^{-n_i x} \int_{\frac{A}{n_i}}^{\frac{s}{n_i}} \left| \int_x^t |t-u|^{2r-1} \, |f^{(m+2r)}(u)| du \right| dt$$

$$\leq n_i \int_{\frac{A}{n_i}}^{\frac{s}{n_i}} |t-x|^{2r-1} \left| \int_x^t \frac{1}{u^r} u^r \, |f^{(m+2r)}(u)| du \right| dt$$

$$\leq n_i \|f^{(m)}\|_{d,2r} \int_{\frac{A}{n_i}}^{\frac{s}{n_i}} |t-x|^{2r} \, (x^{-r} + t^{-r}) dt$$

But $A/n_i \leq t, x \leq s/n_i$ implies that

$$(t-x)^{2r} \le \left(\frac{s-A}{n_i}\right)^{2r}$$

$$t^{-r} \le \left(\frac{A}{n_i}\right)^{-r}$$

$$x^{-r} \le \left(\frac{A}{n_i}\right)^{-r}.$$

Thus we have that the above expression is majorized by

$$M\, n_i^{-r} \| f^{(m)} \|_{d,2r},$$

therefore, by (1.5.1a),

(3.2.14) $|S_o(s,i,x)| \le M\, n^{-r} \| f^{(m)} \|_{d,2r}$.

We still have to consider the terms corresponding to $k \ge 1$ in $S(s,i,x)$. Let

$$S_r(s,i,x) = n_i \sum_{k=1}^{\infty} q_{n_i,k}(x) \int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} \left| \int_x^t |t-u|^{2r-1} |f^{(m+2r)}(u)| du \right| dt$$

$$\le n_i \| u^r f^{(m+2r)} \|_\infty \sum_{k=1}^{\infty} q_{n_i,k}(x) \int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} |t-x|^{2r-1} \left| \int_x^t u^{-r} du \right| dt$$

$$\le n_i \| f^{(m)} \|_{d,2r} \sum_{k=1}^{\infty} q_{n_i,k}(x) \int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} (t-x)^{2r} (x^{-r} + t^{-r}) dt$$

$$\le n_i \| f^{(m)} \|_{d,2r} \frac{1}{x^r} \sum_{k=1}^{\infty} q_{n_i,k}(x) \int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} (t-x)^{2r} dt$$

$$+ n_i \| f^{(m)} \|_{d,2r} \sum_{k=1}^{\infty} q_{n_i,k}(x) \int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} (t-x)^{2r} t^{-r} dt.$$

(3.2.15) $= I_1 + I_2$, say.

Using lemma 3.2.1,

$$I_1 \le n_i \|f^{(m)}\|_{d,2r} \frac{1}{x^r} \sum_{k=0}^{\infty} q_{n_i,k}(x) \int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} (t-x)^{2r} dt$$

$$\le M \|f^{(m)}\|_{d,2r} \frac{1}{x^r} \frac{1}{n_i^{2r}} \sum_{j=0}^{2r} \sum_{k=0}^{\infty} q_{n_i,k}(x).(k-n_i x)^{2r-j}$$

Choosing $A = 1$. By (1.5.1a), we have $x \ge \frac{1}{n_i}$. Thus, by corollary 2.1.1,

$$|I_1| \le M\, n_i^{-r} \|f^{(m)}\|_{d,2r} \frac{1}{(n_i x)^r} \sum_{j=0}^{2r} (n_i x)^{[\frac{2r-j}{2}]}$$

$$\le M\, n^{-r} \|f^{(m)}\|_{d,2r}.$$

Next we take up I_2, we have

$$|I_2| \le n_i \|f^{(m)}\|_{d,2r} \sum_{k=1}^{\infty} q_{n_i,k}(x) \left(\frac{n_i}{k}\right)^r \int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} (t-x)^{2r} dt.$$

$$\le M \|f^{(m)}\|_{d,2r} \int_0^{\infty} n_i \sum_{k=0}^{\infty} q_{n_i,k}(x) \left(\frac{n_i}{k+1}\right)^r (t-x)^{2r} \chi_{k,s,n_i}(t) dt$$

where $\chi_{k,s,n_i}(t)$ is the characteristic function of the interval $[\frac{k}{n_i}, \frac{k+s}{n_i})$. Using Holder's inequality we get

$$|I_2| \le M \|f^{(m)}\|_{d,2r} \left(\int_0^{\infty} n_i \sum_{k=0}^{\infty} q_{n_i,k}(x) \left(\frac{n_i}{k+1}\right)^{2r} \chi_{k,s,n_i}(t) dt\right)^{1/2}$$

$$\left(\int_0^{\infty} n_i \sum_{k=0}^{\infty} q_{n_i,k}(x) (t-x)^{4r} \chi_{k,s,n_i}(t) dt\right)^{1/2}$$

Applying lemma 2.1.2 and lemma 3.2.1,

$$|I_2| \le M \|f^{(m)}\|_{d,2r}\, x^{-r}\left[n_i^{-4r}\sum_{j=0}^{4r}\sum_{k=0}^{\infty} q_{n_i,k}(x)(k-n_i x)^{4r-j}\right]^{1/2}$$

Here onwards a similar analysis as done for I_1 gives

$$|I_2| \le M\, n^{-r}\|f^{(m)}\|_{d,2r}.$$

The estimates of I_1 and I_2 together imply

$$|S_r(s,i,x)| \le M\, n^{-r}\|f^{(m)}\|_{d,2r}.$$

This along with (3.2.14) gives

$$|S(s,i,x)| \le M\, n^{-r}\|f^{(m)}\|_{d,2r}\quad, \qquad x \ge A/n$$

Hence by (3.2.12),

$$|S_{12}(x)| \le M\, n^{-r}\|f^{(m)}\|_{d,2r}\quad, \qquad x \ge A/n\ .$$

Thus completing the analysis for case(i).

Case(ii): Let $x \le A/n$. Using Taylor's expansion with integral form of remainder we have

$$S_1(x) = S_{11}(x) + S_{12}(x)$$

where

$$S_{11}(x) = \frac{1}{(m-1)!}\sum_{s=0}^{m}\binom{m}{s}\sum_{i=0}^{r-1} C(i,r)\, n_i^m \sum_{k=0}^{\infty} q_{n_i,k}(x)$$

$$\cdot\sum_{l=1}^{r-1}\frac{1}{l!}\, f^{(m+l)}(x)\cdot\int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}}\left(\frac{k+s}{n_i} - t\right)^{m-1}\cdot(t-x)^{l}\,dt$$

and

$$S_{12}(x) = \frac{1}{(m-1)!}\,\frac{1}{(r-1)!}\sum_{s=0}^{m}\left[\begin{matrix} m \\ s \end{matrix}\right](-1)^{m-s}\sum_{i=0}^{r-1} C(i,r)\, n_i^m$$

$$\cdot\sum_{k=0}^{\infty} q_{n_i,k}(x)\int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}}\left(\frac{k+s}{n_i} - t\right)^{m-1}\int_x^t (t-u)^{r-1} f^{(m+r)}(u)\,du\,dt.$$

Using lemma 3.2 1, we get as in the previous case

$$S_{11}(x) = \sum_{s=1}^{m}\sum_{l=1}^{r-1}\sum_{\nu=0}^{l} A(m,s,l,\nu) f^{(m+l)}(x)\sum_{i=0}^{r-1} C(i,r)\frac{1}{n_i^l}\, T_{n_i,l-\nu}(x).$$

Since, $1 \le l \le r-1$ and $A(m,s,l,\nu)$ is bounded, by remark 3.2.2,

$$S_{11}(x) = 0 .$$

So all we need is to estimate

$$S_{12}(x) = \frac{1}{(m-1)!}\,\frac{1}{(r-1)!}\sum_{s=0}^{m}\left[\begin{matrix} m \\ s \end{matrix}\right](-1)^{m-s}\sum_{i=0}^{r-1} C(i,r) n_i^m \sum_{k=0}^{\infty} q_{n_i,k}(x)$$

$$\cdot\int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}}\left(\frac{k+s}{n_i} - t\right)^{m-1}\int_x^t (t-u)^{r-1} f^{(m+r)}(u)\,du\,dt.$$

Thus,

$$|S_{12}(x)| \le M \sum_{s=0}^{m}\sum_{i=0}^{r-1} n_i^m \sum_{k=0}^{\infty} q_{n_i,k}(x)$$

$$\cdot\int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}}\left(\frac{k+s}{n_i} - t\right)^{m-1}\left|\int_x^t |t-u|^{r-1}\,|f^{(m+r)}(u)|\,du\right|dt$$

$$\leq M \|f^{(m+r)}\|_\infty \sum_{s=0}^{m} \sum_{i=0}^{r-1} n_i \sum_{k=0}^{\infty} q_{n_i,k}(x) \int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} |t-x|^{r-1} \left|\int_x^t du\right| dt$$

$$\leq M \|f^{(m+r)}\|_\infty \sum_{s=0}^{m} \sum_{i=0}^{r-1} n_i \sum_{k=0}^{\infty} q_{n_i,k}(x) \int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} |t - x|^r dt$$

$$(3.2.17) = M \|f^{(m+r)}\|_\infty \sum_{s=0}^{m} \sum_{i=0}^{r-1} S(s,i,x), \text{ say.}$$

Let $\chi_{k,s,n_i}(t)$ denote the characteristic function of the interval $[\frac{k}{n_i}, \frac{k+s}{n_i})$. Therefore,

$$S(s,i,x) = \int_0^\infty n_i \sum_{k=0}^{\infty} q_{n_i,k}(x) \; |t - x|^r \chi_{k,s,n_i}(t)dt.$$

As a consequence of Holder's inequality

$$S(s,i,x) \leq M \left(\int_0^\infty n_i \sum_{k=0}^{\infty} q_{n_i,k}(x) \; (t - x)^{2r} \chi_{k,s,n_i}(t)dt. \right)^{1/2}$$

Thus applying lemma 3.2.1 and using (1.5.1a),

$$|S(s,i,x)| \leq M \, n^{-r} \left(\sum_{j=0}^{2r} |T_{n_i,2r-j}(x)| \right)^{1/2}$$

We wish to show that

$$(3.2.18) \quad |T_{n_i,2r-j}(x)| \leq M, \qquad 0 \leq j \leq 2r .$$

In view of lemma 2.1.4 this of course is true for $j = 2r-1$ and $2r$, we further have, for $0 \leq j \leq 2r-2$,

$$|T_{n_i,2r-j}(x)| \le \sum_{k=1}^{[\frac{2r-j}{2}]} |a_{k,2r-j}| \, (n_i x)^k$$

$$\le M \sum_{k=1}^{[\frac{2r-j}{2}]} (nx)^k \qquad \text{(by (1.5.1a))} .$$

Now $x \le A/n$ gives (3.2.18) . Therefore,

$$|S(s,i,x)| \le M\, n^{-r}$$

hence by (3.2.17),

$$|S_{12}(x)| \le M\, n^{-r} \| f^{(m)} \|_{d,2r} .$$

The estimates for $S_{11}(x)$ and $S_{12}(x)$ together give

$$|S_1(x)| \le M\, n^{-r} \| f^{(m)} \|_{d,2r} . \qquad x \le A/n .$$

Thus completing case(ii) .

Hence the theorem . ∎

3.3 INVERSE THEOREM

<u>THEOREM</u> <u>3.3.1</u>: Let $f \in C^{(m)}[0,\infty)$, $r \in \mathbb{N}$ and $\phi \in \Phi_{2r}$ then

$$\| S_{n,r}^{(m)} f - f^{(m)} \|_\infty = O(\phi(n^{-1/2}))$$

if and only if $f^{(m)} \in \mathcal{D}_{2r,\phi}$.

<u>Proof</u>: Using lemma 2.1.3 and (3.1.1) we have

$$|S_n^{(m+r)} f(x)| \le M\, n^r \| f^{(m)} \|_\infty , \qquad f \in C^{(m)}[0,\infty)$$

and

$$|S_n^{(m+r)} f(x)| \le M\, \| f^{(m+r)} \|_\infty , \qquad f^{(m)} \in \mathcal{D}_{2r}$$

Therefore,(3.1.2) in conjunction with the lemmas 3.1.1-2 gives

(3.3.1) $\quad \|S_n^{(m)}f\|_{d,2r} \le M\, n^r \|f^{(m)}\|_\infty \qquad\qquad f \in C^{(m)}[0,\infty)$

(3.3.2) $\quad \|S_n^{(m)}f\|_{d,2r} \le M\, \|f^{(m)}\|_{d,2r} \qquad\qquad f^{(m)} \in \mathcal{D}_{2r}$

respectively .

The rest of the proof is same as that of theorem 2.5.1 , hence the details will be omitted .

Suppose $\|S_{n,r}^{(m)}f-f^{(m)}\|_\infty \le C.\phi(n^{-1/2})$. We have

$$K(t^{2r},f^{(m)}) = \inf_{g\in\mathcal{D}_{2r}} \left\{ \|f^{(m)}-g\|_\infty + t^{2r}\|g\|_{d,2r}\right\}$$

$$\le \left\{ \|S_{n,r}^{(m)}f-f^{(m)}\|_\infty + t^{2r}\|S_{n,r}^{(m)}f\|_{d,2r}\right\}$$

$$\le \left\{ C.\phi(n^{-1/2}) + t^{2r}\left\{ \|S_{n,r}^{(m)}(f-g)\|_\infty + \|S_{n,r}^{(m)}g\|_{d,2r} \right\}\right\}$$

$$(g^{(m)} \in \mathcal{D}_{2r})$$

$$\le C_1 \left\{ \phi(n^{-1/2}) + t^{2r}\left\{ n^r\|f^{(m)}-g^{(m)}\|_\infty + \|g^{(m)}\|_{d,2r}\right\}\right\}$$

(by (2.5.10) and (3.3.1-2))

$$\le C_2 \left\{ \phi(n^{-1/2}) + t^{2r}n^rK(n^{-r},f) \right\}$$

Now lemma 1.6.3 gives

$$K(t^{2r},f^{(m)}) = O(\phi(t))$$

in other words $f^{(m)} \in \mathcal{D}_{2r}$.

Conversely suppose $f^{(m)} \in \mathcal{D}_{2r}$. So for given t there exists a g_t such that $g_t^{(m)} \in \mathcal{D}_{2r}$ and

$$\|f^{(m)}-g_t^{(m)}\|_\infty \le C_3\phi(t) ,$$

$$t^{2r}\|g_t^{(m)}\|_{d,2r} \le C_3\phi(t) .$$

This together with (3.1.2) and theorem 3.2.1 gives

$$\|S_{n,r}^{(m)}f-f^{(m)}\|_\infty \le \|S_{n,r}^{(m)}(f-g_t)\|_\infty + \|S_{n,r}^{(m)}g_t-g_t^{(m)}\|_\infty + \|g_t^{(m)}-f^{(m)}\|_\infty$$

$$\le 2.\|f^{(m)}-g_t^{(m)}\|_\infty + M\,n^{-r}\|g_t^{(m)}\|_{d,2r}$$

$$\le C_4 \left\{ \phi(t) + n^{-r}t^{-2r}\phi(t) \right\}$$

From here the converse follows on taking $t = n^{-1/2}$. ■

CHAPTER IV

GLOBAL SIMULTANEOUS APPROXIMATION BY LINEAR COMBINATIONS OF BERNSTEIN POLYNOMIALS

4.1 PRELIMINARIES

The well known Bernstein polynomials are given by

$$B_n(f,x) = B_nf(x) = \sum_{k=0}^{n} p_{n,k}(x)\ f(\tfrac{k}{n}),\ n \in \mathbb{N} \text{ and } f \in C[0,1]$$

where $p_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k}$.

In this section some basic results concerning the Bernstein polynomials and its kernel $p_{n,k}(x)$ are given .

Throughout this chapter and the next X will denote $x(1-x)$.

<u>LEMMA</u> <u>4.1.1</u> [50]: Let $f \in C[0,1]$, $m \in \mathbb{N}_o$, then

$$B_n^{(m)}(f,x) = B_n^{(m)}f(x) = \frac{n!}{(n-m)!} \sum_{k=0}^{n-m} p_{n-m,k}(x)\ \Delta^m_{1/n}f(k/n)$$

where $\Delta_{1/n}f(x) = f(x+1/n) - f(x)$.

For $f \in C^m[0,1]$ by mean value theorem and the above lemma,

$$(4.1.1)\quad B_n^{(m)}f(x) = \frac{n!}{(n-m)!}n^{-m} \sum_{k=0}^{n-m} p_{n-m,k}(x)\ f^{(m)}(\frac{k+m\theta_k}{n}) \quad ,$$

$$0 < \theta_k < 1.$$

<u>LEMMA</u> <u>4.1.2</u> [25]: Let $m \in \mathbb{N}_o$, then

$$\sum_{k=0}^{n} p_{n,k}(x)\frac{n^m}{(k+1)^m} \le \frac{m!}{x^m} \quad , \text{ and}$$

$$\sum_{k=0}^{n} p_{n,k}(x)\frac{n^m}{(n-k+1)^m} \le \frac{m!}{(1-x)^m} \quad .$$

<u>LEMMA</u> <u>4.1.3</u> [25]: For $r \in \mathbb{N}$, $p_{n,k}^{(2r)}(x)$ is a sum of terms of the type

$$q_{l,m}(x)\ \frac{(k-nx)^{2r-2l-m}\ n^{l}}{(x(1-x))^{2r-l}}\ p_{n,k}(x)$$

where $l,m \ge 0$, $2r-2l-m \ge 0$ and $q_{l,m}(x)$ is a polynomial in x that does not depend on n and k.

Next lemma is about the moments of the Bernstein polynomials,

$$\mu_{n,s}(x) = \sum_{k=0}^{n} p_{n,k}(x)\left[\frac{k}{n} - x\right]^{s} , \quad s \in \mathbb{N}_o .$$

Following [20] and [50] we define

$$T_{n,s}(x) = \sum_{k=0}^{n} p_{n,k}(x).(k-nx)^{s}.$$

<u>LEMMA</u> <u>4.1.4</u> [25]: Let $T_{n,s}(x)$ be as defined above then

(4.1.2a) $T_{n,0}(x) \equiv 1$ and $T_{n,1}(x) \equiv 0$.

(4.1.2b) $$T_{n,2s}(x) = c_s\ (nX)^s + \sum_{i=1}^{s-1} c_{i,s}(x)\ (nX)^{s-i} \qquad s \ge 1$$

and

(4.1.2c) $$T_{n,2s+1}(x) = b_s(1-2x)(nX)^s + \sum_{i=1}^{s-1} b_{i,s}(x)\ (nX)^{s-i} \qquad s \ge 1$$

where $c_s, b_s, c_{s,i}(x)$ and $b_{s,i}(x)$ are independent of n. Further (4.1.2b-c) can be combined together to be written as

$$(4.1.3) \qquad T_{n,s}(x) = \sum_{i=1}^{[s/2]} a_{i,s}(x)\,(nX)^i \qquad s\geq 2$$

where $a_{i,s}(x)$ is independent of n and $a_{[s/2],s}(x)$ is a constant.

Corollary 4.1.1: Let $\frac{A}{n} \leq x \leq 1-\frac{A}{n}$, A some constant. Then

$$|T_{n,s}(x)| \leq K.(nX)^{[s/2]} \quad , \; s \geq 2,$$

where K is a constant independent of n .

Proof: Since $\frac{A}{n} \leq x \leq 1-\frac{A}{n}$ we have that $nX.\frac{2}{A} \geq 1$

Hence,

$$|T_{n,s}(x)| \leq \sum_{i=1}^{[s/2]} |a_{i,s}(x)|.(nX)^i \leq K.(nX)^{[s/2]}. \blacksquare$$

The spaces A_{2r}^m and $A_{2r,\phi}^m$

Following Ditzian[25] ,

$$A_{2r} = \left\{ f \in C[0,1] \;\middle|\; \begin{array}{l} f^{(2r-1)} \in \text{loc. A.C.}(0,1) \text{ and} \\ \operatorname*{ess.sup}_{0\leq x\leq 1} |X^r f^{(2r)}(x)| < \infty \end{array} \right\}.$$

Let

$$A_{2r}^m = \left\{ f \in C^m[0,1] \;\middle|\; \sup_{0\leq x\leq 1} |f^{(m+r)}(x)| < \infty \text{ and } f^{(m)} \in A_{2r} \right\}$$

and

$$\|f\|_{m,\infty,2r} = \|f\|_{m,\infty} + \|f^{(m+r)}\|_\infty + \|X^r f^{(m+2r)}\|_\infty \quad , \; f \in A_{2r}^m$$

where $\|f\|_{m,\infty} = \|f\|_\infty + \|f^{(m)}\|_\infty$ and $\|f\|_\infty = \underset{0 \le x \le 1}{\text{ess.sup}} |f(x)|$.

<u>THEOREM</u> <u>4.1.1</u>: The space A^m_{2r} is complete with respect to the norm $\|\cdot\|_{m,\infty,2r}$.

<u>Proof</u>: Suppose $\{f_n\}$ is a Cauchy sequence in $(A^m_{2r} , \|\cdot\|_{m,\infty,2r})$ then $\{f_n\}$ and $\{f_n^{(m+r)}\}$, $\{X^r f_n^{(2r)}\}$ are cauchy in $(C^m[0,1],\|\cdot\|_{m,\infty})$ and $(C[0,1] , \|\cdot\|_\infty)$ respectively . But these spaces being complete we have f and g , h in $C^m[0,1]$ and $C[0,1]$ respectively such that

$\|f_n - f\|_{m,\infty}$ and $\|f_n^{(m+r)} - g\|_\infty$, $\|X^r f_n^{(m+2r)} - h\|_\infty$ go to zero as n goes to infinity .

Now define,

$$\tau_o = D^{m+r} + X^r D^{m+2r} \quad \text{and}$$

$$\tau_1 = D^{m+r} - X^r D^{m+2r} \quad .$$

Following the notations of definition 1.5.5,

$$D(T_{\tau_o,\infty,\infty}) = \left\{ f : f^{(m+2r-1)} \in \text{loc.A.C.}(0,\infty) \text{ and } \|\tau_o f\|_\infty < \infty \right\} \text{ and}$$

$$D(T_{\tau_1,\infty,\infty}) = \left\{ f : f^{(m+2r-1)} \in \text{loc.A.C.}(0,\infty) \text{ and } \|\tau_1 f\|_\infty < \infty \right\} .$$

Clearly $A^m_{2r} = D(T_{\tau_o,\infty,\infty}) \cap D(T_{\tau_1,\infty,\infty})$.

By lemma 1.6.4, $T_{\tau_o,\infty,\infty}$ and $T_{\tau_1,\infty,\infty}$ are closed.

Also $T_{\tau_o,\infty,\infty} f_n$ converges to $g+h$ and $T_{\tau_1,\infty,\infty} f_n$ converges to $g-h$ in the sup norm. Thus the closedness of the operators imply

$f \in D(T_{\tau_o,\infty,\infty}) \cap D(T_{\tau_1,\infty,\infty})$, $T_{\tau_o,\infty,\infty}f = g + h$ and

$T_{\tau_1,\infty,\infty}f = g - h$.

From here it follows that $f \in A^m_{2r}$, $g = f^{(m+r)}$ and $h = X^r f^{(m+2r)}$.

Hence the theorem . ■

The space $A^m_{2r,\phi}$ is defined as follows

$$A^m_{2r,\phi} = \left\{ f \in C^m[0,1]: K_m(t^{2r},f) = O(\phi(t)) \right\}$$

where $\phi \in \Phi_{2r}$, and

$$K_m(t,f) = \inf_{g \in A^m_{2r}} \left\{ \|f-g\|_{m,\infty} + t\,\|g\|_{m,\infty,2r} \right\}$$

<u>LEMMA</u> <u>4.1.5</u>[25]: Suppose $X^r f^{(2r)} \in L_\infty[0,1]$ and $f^{(2r-1)} \in A.C.[\alpha,\beta]$ for all α, β, $0 < \alpha < \beta < 1$. Then, for $m = 1, \ldots, r-1$

$$|X^{r-m} f^{(2r-m)}(x)| \le B(m)(\|f\|_\infty + \|X^r f^{(2r)}\|_\infty).$$

The next lemma is a simple consequence of lemma 1.6.1.

<u>LEMMA</u> <u>4.1.6</u>: Let $f \in A^m_{2r}$. Then there exists a constant M independent of f such that for $i = 1,\ldots,r$

$$\|f^{(m+i)}\|_\infty \le M\|f^{(m)}\|_\infty + \|f^{(m+r)}\|_\infty.$$

<u>LEMMA</u> <u>4.1.7</u>: Let $m,r \in \mathbb{N}_o$, then $A^m_{2r} \subseteq A^o_{2r} \subseteq A_{2r}$ moreover

$$\|f\|_\infty + \|X^r f^{(2r)}\|_\infty \le \|f\|_{\infty,2r} \le M\,\|f\|_{m,\infty,2r} , \qquad \|f\|_{\infty,2r} \equiv \|f\|_{0,\infty,2r} .$$

<u>Proof</u>: The first part of the inequality and that $A^o_{2r} \subseteq A_{2r}$ is obvious , in fact the containment is proper , this could be seen through $f(x) = x^r \log x$.

As for $\mathcal{A}^m_{2r} \subseteq \mathcal{A}^o_{2r}$ and the second part of the inequality we proceed as follows .

For $m = r = 0$ the result is trivial . So let m , $r \geq 1$, $f \in \mathcal{A}^m_{2r}$ implies that $f^{(m+2r-1)} \in$ loc A.C.(0,1) , $\|f^{(m+r)}\|_\infty < \infty$ and $\|X^r f^{(m+2r)}\|_\infty < \infty$. In order to show that $f \in \mathcal{A}^o_{2r}$ we must have that $f^{(2r-1)} \in$ loc A.C.(0,1) , which ofcourse is true , and $\|f^{(r)}\|_\infty$, $\|X^r f^{(2r)}\|_\infty$ is finite .

By lemma 1.6.1,

$$\|f^{(r)}\|_\infty \leq M(\|f\|_\infty + \|f^{(m+r)}\|_\infty) < \infty$$

To show that $\|X^r f^{(2r)}\|_\infty < \infty$ consider the cases (i) $m \geq r$ and (ii) $1 \leq m < r$.

Let $m \geq r$, then $m+r \geq 2r$ therefore by lemma 1.6.1,

$$\|f^{(2r)}\|_\infty \leq M(\|f\|_\infty + \|f^{(m+r)}\|_\infty)$$

moreover , since $X \leq \frac{1}{4}$,

$$\|X^r f^{(2r)}\|_\infty \leq \left(\frac{1}{4}\right)^r \|f^{(2r)}\|_\infty \leq M(\|f\|_\infty + \|f^{(m+r)}\|_\infty) .$$

Consequently ,

$$\|f\|_{\infty,2r} \leq M \|f\|_{m,\infty,2r} .$$

Next , let $1 \leq m < r$, in this case applying lemma 4.1.5 ,

$$\|X^r f^{(2r)}\|_\infty \leq \left(\frac{1}{4}\right)^m \|X^{r-m} f^{(m+2r-m)}\|_\infty \leq M(\|f^{(m)}\|_\infty + \|X^r f^{(m+2r)}\|_\infty) .$$

Therefore ,

$$\|f\|_{\infty,2r} \leq M \|f\|_{m,\infty,2r} .$$

Hence the lemma . ∎

4.2 BERNSTEIN TYPE INEQUALITIES

In this section some Bernstein type inequalities are established these will be useful in proving the inverse theorem.

LEMMA 4.2.1 Let $f \in C^m[0,1]$ and $r \in \mathbb{N}_o$. Then,

$$(4.2.1) \qquad \|X^r B_n^{(m+2r)} f\|_\infty \le M\, n^r \|f^{(m)}\|_\infty .$$

Proof: Since $B_n f(x)$ is a a polynomial of degree n, $B_n^{(m+2r)} f(x)$ is zero for $n < m+2r$, hence,(4.2.1) holds trivially for such values of n.

Let $n \ge m + 2r$.

Also for r = 0 (4.2.1) follows from (4.1.1). For $r \ge 1$ we consider two cases.

Case (i): Let $\min(x,1-x) \le A/(n-m)$, A some constant .

Using lemma 4.1.1,

$$|X^r B_n^{(m+2r)} f(x)| \le X^r \frac{n!}{(n-m-2r)!} \sum_{k=0}^{n-m-2r} p_{n-m-2r,k}(x) |\Delta_{1/n}^{m+2r} f(\tfrac{k}{n})|$$

Now, since $f \in C^m[0,1)$, by mean value theorem ,

$$|\Delta_{1/n}^{m+2r} f(\tfrac{k}{n})| = \frac{1}{n^m} |\Delta_{1/n}^{2r} f^{(m)}(\frac{k+m\theta_k}{n})|, \quad 0 < \theta_k < 1$$

$$\le \frac{1}{n^m} \sum_{j=0}^{2r} \binom{2r}{j} |f^{(m)}(\frac{k+m\theta_k+j}{n})|$$

$$\le \frac{1}{n^m}\, 2^{2r} \|f^{(m)}\|_\infty$$

Consequently,

$$|X^r B_n^{(m+2r)} f(x)| \le M\, n^{2r}\, X^r\, 2^{2r} \|f^{(m)}\|_\infty$$
$$\le M\, n^r ((n-m)X)^r\, \|f^{(m)}\|_\infty$$

But, min $(x,\ 1-x) \le A/(n-m)$ implies that $((n-m)X) \le A$. Hence

(4.2.2) $$|X^r B_n^{(m+2r)} f(x)| \le M\, n^r\, \|f^{(m)}\|_\infty\ , \quad \min(\ x,\ 1-x) \le \frac{A}{n-m}.$$

<u>Case (ii)</u>. Let $A/(n-m) \le x \le 1 - A/(n-m)$.

Using (4.1.1) and mean value theorem,

$$X^r B_n^{(m+2r)} f(x) = X^r \frac{n!}{(n-m)!} \frac{1}{n^m} \sum_{k=0}^{n-m} p_{n-m,k}^{(2r)}(x)\ f^{(m)}\Big(\frac{k+m\theta_k}{n}\Big)$$
$$(0<\theta_k<1)$$

Therefore by lemma 4.1.3, $X^r B_n^{(m+2r)} f(x)$ is a sum of terms of the type,

$$S(l,s,x) = q_{l,s}(x)\ X^r \frac{n!}{(n-m)!} \frac{1}{n^m} \frac{1}{X^{2r-l}} (n-m)^l .$$
$$\cdot \sum_{k=0}^{n-m} (k-(n-m)x)^{2r-2l-s} p_{n-m,k}(x)\ f^{(m)}\Big(\frac{k+m\theta_k}{n}\Big)$$

where $l,\ s \ge 0$, $2r-2l-s \ge 0$ and $q_{l,s}(x)$ is a polynomial independent of n-m and k.

Thus,

$$|S(l,s,x)| \le M\, |q_{l,s}(x)|\, \frac{1}{X^{r-l}} (n-m)^l\, \|f^{(m)}\|_\infty .$$
$$\sum_{k=0}^{n-m} |k-(n-m)x|^{2r-2l-s} p_{n-m,k}(x)$$

Further, since $q_{l,s}(s)$ is a polynomial independent of n-m and k, and $A/n-m \le x \le 1-A/n-m$ we have that $|q_{l,s}(x)| \le K$, where K is

a constant independent of n. Therefore , by Cauchy-Schwarz inequality ,

$$|S(l,s,x)| \le M \frac{1}{X^{r-l}} (n-m)^{l} \|f^{(m)}\|_{\infty} \cdot \left[\sum_{k=0}^{n-m} (k-(n-m)x)^{4r-4l-2s} p_{n-m,k}(x) \right]^{1/2}$$

$$= M \|f^{(m)}\|_{\infty} \frac{(n-m)^{l}}{X^{r-l}} \left[T_{n-m,4r-4l-2s}(x) \right]^{1/2}$$

Now the choice of x allows to apply the corollary 4.1.1 and get

$$|S(l,s,x)| \le M \|f^{(m)}\|_{\infty} \frac{(n-m)^{l}}{X^{r-l}} (M((n-m)X)^{2r-2l-s})^{1/2}$$

$$\le M \|f^{(m)}\|_{\infty} (n-m)^{r} \frac{1}{((n-m)X)^{s/2}}$$

$$\le M n^{r} \|f^{(m)}\|_{\infty} \tag{4.2.3}$$

The last inequality is a consequence of $A/(n-m) \le x \le 1-A/(n-m)$ and $s \ge 0$.

Since $X^{r} B_{n}^{(m+2r)} f(x)$ is a sum of terms of the type $S(l,s,x)$, (4.2.3) implies

$$|X^{r} B_{n}^{(m+2r)} f(x)| \le M n^{r} \|f^{(m)}\|_{\infty} , \quad \frac{A}{n-m} \le x \le 1 - \frac{A}{n-m} \tag{4.2.4}$$

Inequalities (4.2.2) and (4.2.4) imply

$$|X^{r} B_{n}^{(m+2r)} f(x)| \le M n^{r} \|f^{(m)}\|_{\infty} , \qquad x \in [0,1]$$

and hence the lemma. ■

<u>Corollary 4.2.1</u>: Let $f \in C^{m}[0,1]$ and $r \in \mathbb{N}_{o}$. Then

$$\|B_{n}^{(m)} f\|_{\infty,2r} \le M n^{r} \|f^{(m)}\|_{\infty} ,$$

where $\|f\|_{\infty,2r} = \|f\|_{\infty} + \|f^{(r)}\|_{\infty} + \|X^{r} f^{(2r)}\|_{\infty}$.

Proof: By (4.1.1),

$$\|B_n^{(m)} f\| \le M \, \|f^{(m)}\|_\infty \, . \tag{4.2.5}$$

Also ,

$$|B_n^{(m+r)} f(x)| \le \frac{n!}{(n-m-r)!} \, \frac{1}{n^m} \sum_{k=0}^{n-m-r} p_{n-m-r,k}(x) \; |\Delta_{1/n}^{r} f^{(m)}(\frac{k+m\theta_k}{n})|$$

$$0 < \theta_k < 1$$

$$\le M \, n^r \|f^{(m)}\|_\infty .$$

Now, the corollary follows from lemma 4.2.1 . ∎

LEMMA 4.2.2: Let $f \in A_{2r}^{m}$. Then ,

$$\|X^r B_n^{(m+2r)} f(x)\|_\infty \le M \, \|X^r f^{(m+2r)}\|_\infty$$

Proof: Since $B_n f(x)$ is a polynomial of degree n , for $n < m+2r$, the result is trivial . So let $n \ge m+2r$.

By mean value theorem,

$$n^{m+2r} \, |\Delta_{1/2}^{m+2r} f(\frac{k}{n})| \le \sup_{\frac{k}{n} < \xi < \frac{k+m+2r}{n}} |f^{(m+2r)}(\xi)|$$

$$\le \|(\xi(1-\xi))^r f^{(m+2r)}\|_\infty \sup_{\frac{k}{n} < \xi < \frac{k+m+2r}{n}} (\xi(1-\xi))^r$$

Hence, for $1 \le k \le n-m-2r-1$,

$$n^{m+2r} |\Delta_{1/n}^{m+2r} f(\frac{k}{n})| \le (\frac{n}{k})^r \; (\frac{n}{n-k-m-2r})^r \; \|X^r f^{(m+2r)}\|_\infty$$

$$\le M \, (\frac{n-m-2r}{k+1})^r \; (\frac{n-m-2r}{n-m-2r-k+1})^r \; \|X^r f^{(m+2r)}\|_\infty \tag{4.2.6}$$

Further from (2.3.2),

$$\Delta_{1/n}^{m+2r} f(0) = I + \sum_{l=1}^{2r+m-1} S_l$$

where

$$I = \frac{1}{(2r+m-1)!} \sum_{j=1}^{2r+m} \binom{2r+m}{j} (-1)^j \int_0^{\frac{j}{n}} u^{2r+m-1} f^{(m+2r)}(u)du$$

and

$$S_l = \frac{(-1)^{l+1}}{l!} \frac{1}{n^l} \sum_{j=1}^{2r+m} \binom{2r+m}{j} (-1)^j j^l f^{(l)}(j/n).$$

Now,

$$|I| \le \frac{1}{(2r+m-1)!} \sum_{j=1}^{2r+m} \binom{2r+m}{j} \int_0^{\frac{2r+m}{n}} u^{r+m-1} \frac{u^r(1-u)^r}{(1-u)^r} |f^{(m+2r)}(u)|du$$

$$\le M \|U^r f^{(m+2r)}\|_\infty \int_0^{(2r+m)/n} \frac{u^{r+m-1}}{(1-u)^r} du \quad , \qquad (U = u(1-u)) .$$

(4.2.7) $$\le M \frac{1}{n^{m+r}} \|U^r f^{(m+2r)}\|_\infty$$

and from (2.3.4),

$$|S_l| \le M \frac{1}{n^l} \sum_{i=0}^{l-1} K. \sum_{0\le j\le l-i-1} |\Delta_{1/n}^{2r+m-l} f^{(l)}(\frac{i+j+1}{n})|$$

(K is a constant independent of n and f).

Therefore, by mean value theorem,

$$n^{2r+m-l} |\Delta^{2r+m-l} f^{(l)}(\frac{i+j+1}{n})|$$

$$\le \sup_{\frac{i+j+1}{n} < \xi < \frac{i+j+1+2r+m-l}{n}} |f^{(m+2r)}(\xi)|$$

$$\le M\ n^r . \| (\xi(1-\xi))^r\ f^{(m+2r)} \|_\infty$$

Thus,

(4.2.8) $$|S_2| \le M\ \frac{1}{n^{r+m}} \|X^r f^{(m+2r)}\|_\infty$$

Inequalities (4.2.7-8) imply

(4.2.9a) $$n^{m+2r} |\Delta_{1/n}^{2r+m} f(0)| \le M\ n^r\ \|X^r f^{(m+2r)}\|_\infty .$$

Similarly, for $k = n-m-2r$,

(4.2.9b) $$n^{m+2r} |\Delta_{1/n}^{2r+m} f(\frac{n-m-2r}{n})| \le M\ n^r\ \|X^r f^{(m+2r)}\|_\infty$$

Now, using (4.2.6) and (4.2.9a-b), one gets,

$$X^r |B_n^{(m+2r)} f(x)| \le n^{m+2r}\ X^r \sum_{k=0}^{n-m-2r} p_{n-m-2r,k}(x) |\Delta_{1/n}^{2r+m}\ f(\frac{k}{n})|$$

$$\le M\ \|X^r f^{(m+2r)}\|_\infty\ X^r \sum_{k=0}^{n-m-2r} p_{n-m-2r,k}(x) \left[\frac{n-m-2r}{k+1} \right]^r \left[\frac{n-m-2r}{n-m-2r-k+1} \right]^r$$

$$\le M\ \|X^r f^{(m+2r)}\|_\infty\ X^r \left(\sum_{k=0}^{n-m-2r} p_{n-m-2r,k}(x) \left[\frac{n-m-2r}{k+1} \right]^r + \sum_{k=0}^{n-m-2r} p_{n-m-2r,k}(x) \left[\frac{n-m-2r}{n-m-2r-k+1} \right]^r \right)$$

Therefore by lemma 4.1.2,

$$X^r |B_n^{(m+2r)} f(x)| \le M\ \|X^r f^{(m+2r)}\|_\infty X^r . \left[\frac{r!}{x^r} + \frac{r!}{(1-x)^r} \right]$$

$$\le M\ \|X^r f^{(m+2r)}\|_\infty$$

Hence the lemma. ∎

Corollary 4.2.2: Let $f \in A^m_{2r}$, then

$$\|B_n^{(m)} f\|_{\infty,2r} \le M \, \|f^{(m)}\|_{\infty,2r}.$$

Proof: As an application of mean value theorem and lemma 4.1.1 ,

$$|B_n^{(m+r)} f(x)| \le M \, \|f^{(m+r)}\|_{\infty} \quad , \quad f \in A^m_{2r}$$

This together with (4.2.5) and lemma 4.2.2 establishes the corollary.■

4.3 ORDER OF APPROXIMATION FOR $f \in A^m_{2r}$

In this section an order of approximation result for simultaneous approximation by linear combinations of Bernstein polynomials , for $f \in A^m_{2r}$, will be established .

LEMMA 4.3.1: Let $k,s,m \in \mathbb{N}_o$, $n \in \mathbb{N}$ and $x \in \mathbb{R}$, then

$$(k-nx)^s = \sum_{j=0}^{s} c_{j,s}(x) \, (k-(n-m)x)^j$$

where $c_{j,s}(x) = \binom{s}{j} (-mx)^{s-j}$ is a polynomial independent of n and k.

Proof: $(k-nx)^s = (k-(n-m)x-mx)^s$

$$= \sum_{j=0}^{s} \binom{s}{j} (k-(n-m)x)^j \, (-mx)^{s-j}$$

$$= \sum_{j=0}^{s} c_{j,s}(x) \, (k-(n-m)x)^j \qquad ■$$

<u>LEMMA</u> <u>4.3.2</u>: Let $m \in \mathbb{N}_o$ and $n \in \mathbb{N}$, then

$$\frac{n!}{(n-m)!n^m} = 1 + \sum_{k=1}^{m-1} b_k \frac{1}{n^k} = \sum_{k=0}^{m-1} b_k \frac{1}{n^k}$$

<u>Proof</u>
$$\frac{n!}{(n-m)!n^m} = \frac{n(n-1) - (n-m+1)}{n^m}$$

$$= (1 - \frac{1}{n})(1 - \frac{2}{n}) \ldots (1 - \frac{(m-1)}{n})$$

$$= 1 + \sum_{k=1}^{m-1} b_k \frac{1}{n^k} = \sum_{k=0}^{m-1} b_k \frac{1}{n^k} \qquad \blacksquare$$

<u>LEMMA</u> <u>4.3.3</u>: Let $m, s \in \mathbb{N}_o$, $n \in \mathbb{N}$, $s \leq m$ and

$$J_n(x) = n \sum_{k=0}^{n-m} p_{n-m,k}(x) \int_{\frac{k}{n}}^{\frac{k+s}{n}} |t-x|^r dt$$

Then

$$|J_n(x)| \leq M \frac{1}{n^r} \left[\sum_{\nu=0}^{2r} \sum_{j=0}^{2r-\nu} |T_{n-m,j}(x)| \right]^{1/2}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \text{ and } T_{n,s}(x) = \sum_{k=0}^{n} p_{n,k}(x) (k-nx)^s$$

<u>Proof</u>:

$$J_n(x) = \int_0^1 n \sum_{k=0}^{n-m} p_{n-m,k}(x) \chi_{k,s,n}(t) |t-x|^r dt$$

where $\chi_{k,s,n}(t) = \chi_{[\frac{k}{n}, \frac{k+s}{n}]}(t)$ is the characteristic function of the interval $[\frac{k}{n}, \frac{k+s}{n}]$.

Hence , using Holder's inequality ,

$$J_n(x) \le \left[\int_0^1 n \sum_{k=0}^{n-m} p_{n-m,k}(x)\, \chi_{k,s,n}(t)dt \right]^{1/2}$$

$$\left[\int_0^1 n \sum_{k=0}^{n-m} p_{n-m,k}(x)(t-x)^{2r} \chi_{k,s,n}(t)dt \right]^{1/2}$$

$$(4.3.1) \qquad \le M \left[n \sum_{k=0}^{n-m} p_{n-m,k}(x) \int_{\frac{k}{n}}^{\frac{k+s}{n}} (t-x)^{2r}dt \right]^{1/2}$$

By lemma 3.2.1 and lemma 4.3.1,

$$J_n(x) \le M \left[n \frac{1}{n^{2r+1}} \sum_{\nu=0}^{2r} \frac{1}{(1+\nu)!} \frac{(2r)!}{(2r-\nu)!} s^{1+\nu} \cdot \sum_{k=0}^{n-m} p_{n-m,k}(x)(k-nx)^{2r-\nu} \right]^{1/2}$$

$$\le M \frac{1}{n^r} \left[\sum_{\nu=0}^{2r} \sum_{j=0}^{2r-\nu} c_{j,2r-\nu}(x) \sum_{k=0}^{n-m} p_{n-m,k}(x)(k-(n-m)x)^j \right]^{1/2}$$

Thus ,

$$|J_n(x)|^2 \le M \frac{1}{n^{2r}} \sum_{\nu=0}^{2r} \sum_{j=0}^{2r-\nu} |T_{n-m,j}(x)| .$$

Hence the lemma. ∎

<u>LEMMA</u> <u>4.3.4</u>: Let $m, \alpha, 2\beta \in \mathbb{N}_o$, $n \in \mathbb{N}$, $n > m$, $0 \le s \le m$ and

$$H_n(x) = n \sum_{k=1}^{n-m} p_{n-m,k}(x) \int_{\frac{k}{n}}^{\frac{k+s}{n}} (X^{-\beta} + T^{-\beta}) \, |t-x|^{\alpha} dt$$

where $k < n-m$ when $s = m$,$\alpha \le 2\beta$, $X = x(1-x)$ and $T = t(1-t)$.

Then , for $\frac{A}{n} \le x \le 1 - \frac{A}{n}$,

$$|H_n(x)| \le M \frac{1}{n^{\alpha-\beta}}$$

<u>Proof</u>:

$$H_n(x) = \frac{1}{X^{\beta}} .n \sum_{k=1}^{n-m} p_{n-m,k}(x) \int_{\frac{k}{n}}^{\frac{k+s}{n}} |t-x|^{\alpha} dt$$

$$+ n \sum_{k=1}^{n-m} p_{n-m,k}(x) \int_{\frac{k}{n}}^{\frac{k+s}{n}} T^{-\beta}.|t-x|^{\alpha} dt.$$

$$= I_{1,n}(x) + I_{2,n}(x).$$

Using lemma 4.3.3,

$$|I_{1,n}(x)| \le M X^{-\beta}\frac{1}{n^{\alpha}} \Big(\sum_{\nu=0}^{2\alpha} \sum_{j=0}^{2\alpha-\nu} |T_{n-m,j}(x)| \Big)^{1/2}$$

(4.3.2) $$= M \frac{1}{n^{\alpha-\beta}} \Big(\sum_{\nu=0}^{2\alpha} \sum_{j=0}^{2\alpha-\nu} \frac{1}{(nX)^{2\beta}} |T_{n-m,j}(x)| \Big)^{1/2}$$

In the following it will be shown that for $A/n \le x \le 1-A/n$,

(4.3.3) $$\frac{1}{(nX)^{2\beta}} |T_{n-m,j}(x)| \le M , \qquad 0 \le j \le 2\alpha-\nu , 0 \le \nu \le 2\alpha$$

The choice of x implies $\frac{1}{nX} \le \frac{2}{A}$. Therefore, in view of (4.1.2a),

for j = 0 and 1 , (4.3.3) holds good .

For $j \ge 2$, from (4.1.3),

$$T_{n-m,j}(x) = \sum_{i=1}^{[j/2]} a_{i,j}(x) ((n-m)X)^{i}$$

$a_{i,j}(x)$ being a polynomial independent of n-m is bounded , thus ,

$$\frac{1}{(nX)^{2\beta}} |T_{n-m,j}(x)| \le M \frac{1}{(nX)^{2\beta}} \sum_{i=1}^{[j/2]} ((n-m)X)^i$$

$$\le M \sum_{i=1}^{[j/2]} \frac{(nX)^i}{(nX)^{2\beta}}$$

$1 \le i \le [j/2] \le \alpha$ and $\alpha \le 2\beta$ implies

$$\frac{1}{(nX)^{2\beta-i}} \le \left[\frac{2}{A} \right]^{2\beta-i} , \quad A/n \le x \le 1-A/n.$$

Thus establishing (4.3.3) for $j \ge 2$.
Consequently,

$$(4.3.4) \quad |I_{1,n}(x)| \le M \frac{1}{n^{\alpha-\beta}}$$

Next we take up $I_{2,n}(x)$. For $0 < t < 1$,

$1 \le 2^{p-1}(t^p+(1-t)^p)$, $1 \le p < \infty$ and $1 \le t^p+(1-t)^p$ if $0 \le p < 1$.

Thus ,

$$I_{2,n}(x) \le M\ n \sum_{k=1}^{n-m} p_{n-m,k}(x) \int_{\frac{k}{n}}^{\frac{k+s}{n}} |t-x|^\alpha \left(\frac{1}{t^\beta} + \frac{1}{(1-t)^\beta}\right) dt$$

$$= M\ n \sum_{k=1}^{n-m} p_{n-m,k}(x) \int_{\frac{k}{n}}^{\frac{k+s}{n}} |t-x|^\alpha \frac{1}{t^\beta} dt$$

$$+ M\ n \sum_{k=1}^{n-m} p_{n-m,k}(x) \int_{\frac{k}{n}}^{\frac{k+s}{n}} |t-x|^\alpha \frac{1}{(1-t)^\beta} dt$$

$$= A_1(x) + A_2(x).$$

It is intended to show that

(4.3.5a) $$|A_2(x)| \le M \frac{1}{n^{\alpha}} \frac{1}{(1-x)^{\beta}} \left[\sum_{\nu=0}^{2\alpha} \sum_{j=0}^{2\alpha-\nu} |T_{n-m,j}(x)| \right]^{1/2}$$

and in a similar fashion it can also be shown that

(4.3.5b) $$|A_1(x)| \le M \frac{1}{n^{\alpha}} \frac{1}{x^{\beta}} \left[\sum_{\nu=0}^{2\alpha} \sum_{j=0}^{2\alpha-\nu} |T_{n-m,j}(x)| \right]^{1/2}$$

Now, (4.3.5a-b) together imply

$$|I_{2,n}(x)| \le M \frac{1}{X^{\beta}} \frac{1}{n^{\alpha}} \left[\sum_{\nu=0}^{2\alpha} \sum_{j=0}^{2\alpha-\nu} |T_{n-m,j}(x)| \right]^{1/2}$$

$$\le M \frac{1}{n^{\alpha-\beta}} \left[\sum_{\nu=0}^{2\alpha} \sum_{j=0}^{2\alpha-\nu} \frac{1}{(nX)^{2\beta}} |T_{n-m,j}(x)| \right]^{1/2}$$

which is same as (4.3.2), and hence,

(4.3.6) $$|I_{2,n}(x)| \le M \frac{1}{n^{\alpha-\beta}}$$

the estimates in (4.3.4) and (4.3.6) together give the lemma .

Hence all that is needed is (4.3.5a) .

Now, $k < n - m$ when $s = m$ and $s \le m$ imply $k+s < n$, so

$$A_2(x) \le M\, n \sum_{k=1}^{n-m} p_{n-m,k}(x) \left(\frac{n}{n-k-s}\right)^{\beta} \int_{\frac{k}{n}}^{\frac{k+s}{n}} |t-x|^{\alpha}\, dt$$

$$\le M\, n \sum_{k=0}^{n-m} p_{n-m,k}(x) \left(\frac{n-m}{n-m-k+1}\right)^{\beta} \int_{\frac{k}{n}}^{\frac{k+s}{n}} |t-x|^{\alpha}\, dt$$

$$= M \int_0^1 n \sum_{k=0}^{n-m} p_{n-m,k}(x) \left(\frac{n-m}{n-m-k+1}\right)^{\beta} \chi_{k,s,n}(t) |t-x|^{\alpha}\, dt$$

where $\chi_{k,s,n}(t)$ denotes the characteristic function of the interval $[\frac{k}{n}, \frac{k+s}{n}]$.

Now, using Holder's inequality , the right hand side of the above can be shown not to exceed

$$M\left(\int_0^1 n \sum_{k=0}^{n-m} p_{n-m,k}(x) \left(\frac{n-m}{n-m-k+1}\right)^{2\beta} \chi_{k,s,n}(t)dt\right)^{1/2}$$

$$\left[\int_0^1 n \sum_{k=0}^{n-m} p_{n-m,k}(x)\, \chi_{k,s,n}(t)(t-x)^{2\alpha}dt\right]^{1/2}$$

Therefore, by lemma 4.1.2,

$$A_2(x) \le M\left(\frac{(2\beta)!}{(1-x)^{2\beta}}\right)^{1/2}\left(n \sum_{k=0}^{n-m} p_{n-m,k}(x) \int_{\frac{k}{n}}^{\frac{k+s}{n}} (t-x)^{2\alpha}dt\right)^{1/2}$$

observe that the right hand side of the above is same as that of (4.3.1) except for the factor $\left(\frac{(2\beta)!}{(1-x)^{2\beta}}\right)^{1/2}$, thus the inequality (4.3.5a) now follows in a similar fashion as the previous lemma.

Hence the lemma . ■

<u>LEMMA</u> <u>4.3.5</u>: Let $x \in [0,1]$, $n \in \mathbb{N}$ and $l \in \mathbb{N}_o$ then

$$x^n(1-x)^l \le M\, n^{-l} \quad \text{and}$$

$$(1-x)^n x^l \le M\, n^{-l}$$

<u>Proof</u>: Since ,

$$(1-x+x)^{n+l} = \sum_{k=0}^{n+l} \binom{n+l}{k} (1-x)^k x^{n+l-k} = 1$$

Taking $k = l$,

$$x^n(1-x)^l \le \frac{l!\,n!}{(n+l)!} \le M\, n^{-l} .$$

The other inequality follows on taking $k = n$. ■

THEOREM 4.3.1: Let $f \in A^m_{2r}$ then

$$\|B^{(m)}_{n,r}f - f^{(m)}\|_\infty \le M \frac{1}{n^r} \|f^{(m)}\|_{\infty,2r}$$

Proof: Let $S(x) = B^{(m)}_{n,r}f(x) - f^{(m)}(x)$

$$= \sum_{i=0}^{r-1} C(i,r)\left[B^{(m)}_{n_i}f(x) - f^{(m)}(x) \right]$$

By lemmas 4.1.1 and 3.2.3,

$$S(x) = \sum_{i=0}^{r-1} C(i,r)\left[\frac{n_i!}{(n_i-m)!}\sum_{k=0}^{n_i-m}\left(\frac{1}{(m-1)!}\sum_{s=0}^{m}\binom{m}{s}(-1)^{m-s}\right.\right.$$

$$\left.\left.\int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} \left(\frac{k+s}{n_i} - t\right)^{m-1} f^{(m)}(t)dt\right) p_{n_i-m,k}(x) - f^{(m)}(x)\right]$$

$$= \sum_{i=0}^{r-1} C(i,r)\left[\frac{1}{(m-1)!}\frac{n_i!}{(n_i-m)!}\sum_{s=0}^{m}\binom{m}{s}(-1)^{m-s}\right.$$

$$\left.\cdot\sum_{k=0}^{n_i-m} p_{n_i-m,k}(x)\int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}}\left(\frac{k+s}{n_i} - t\right)^{m-1}\left[f^{(m)}(t)-f^{(m)}(x)\right]dt\right]$$

$$+ \sum_{i=0}^{r-1} C(i,r)\left[\frac{1}{(m-1)!}\frac{n_i!}{(n_i-m)!}\sum_{s=0}^{m}\binom{m}{s}(-1)^{m-s}\right.$$

$$\left.\cdot\sum_{k=0}^{n_i-m} p_{n_i-m,k}(x)\int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}}\left(\frac{k+s}{n_i} - t\right)^{m-1}dt - 1\right] f^{(m)}(x).$$

(4.3.7) $= \quad S_1(x) + S_2(x)$

Using lemma 3.2.1 and (3.2.1) together with the fact that

$\sum_{k=0}^{n} p_{n,k}(x) = 1$, it can be shown that

$$S_2(x) = \sum_{i=0}^{r-1} C(i,r) \left[\frac{n_i!}{(n_i-m)!}n_i^{-m} - 1 \right] f^{(m)}(x)$$

Now by lemma 4.3.2 and (1.5.1d) ,

$$S_2(x) = \sum_{k=r}^{m-1} b_k \sum_{i=0}^{r-1} C(i,r) \frac{1}{n_i^k} f^{(m)}(x) \text{ , provided } m-1 \geq r.$$

Thus ,

$$|S_2(x)| \leq M \frac{1}{n^r} \|f^{(m)}\|_\infty$$

Next we take up $S_1(x)$. The analysis for this part will be carried out in two separate cases , (a) $A/n \leq x \leq 1-A/n$ and (b) $\min(x,1-x) \leq A/n$ where A is some constant which will be specified when required. Before we proceed further let us observe that in $S_1(x)$ the term corresponding to $s = 0$ is identically zero. Therefore, hence forward $1 \leq s \leq m$.

<u>Case (a)</u>: $A/n \leq x \leq 1 - A/n$

Note that the choice of x implies

(4.3.8) $\frac{1}{nX} \leq \frac{2}{A}$

Taylor's expansion for $f^{(m)}(t)$ with integral form of remainder gives

$$S_1(x) = \frac{1}{(m-1)!}\sum_{s=1}^{m} \binom{m}{s} (-1)^{m-s} \sum_{l=1}^{2r-1} \frac{1}{l!}\, f^{(m+l)}(x) \sum_{i=0}^{r-1} C(i,r) \frac{n_i!}{(n_i-m)!}$$

$$\sum_{k=0}^{n_i-m} p_{n_i-m,k}(x) \int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} \left(\frac{k+s}{n_i} - t\right)^{m-1} (t-x)^l dt$$

$$+ \frac{1}{(m-1)!}\,\frac{1}{(2r-1)!} \sum_{s=1}^{m} \binom{m}{s} (-1)^{m-s} \sum_{i=0}^{r-1} C(i,r) \frac{n_i!}{(n_i-m)!}$$

$$\sum_{k=0}^{n_i-m} p_{n_i-m,k}(x) \int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} \left(\frac{k+s}{n_i}-t\right)^{m-1} \int_x^t (t-u)^{2r-1} f^{(m+2r)}(u)\,du\,dt$$

$$(4.3.9) \qquad = S_{11}(x) + S_{12}(x)$$

Applying lemma 3.2.1 with $a = k/n_i$ and $h = s/n_i$ and lemmas 4.3.1 and 4.3.2 , one gets

$$S_{11}(x) = \sum_{s=1}^{m} \binom{m}{s} (-1)^{m-s} \sum_{l=1}^{2r-1} \sum_{\nu=0}^{l} \frac{1}{(m+\nu)!}\,\frac{1}{(l-\nu)!}\, s^{m+\nu}\, f^{(m+l)}(x)$$

$$\sum_{i=0}^{r-1} C(i,r) \frac{n_i!}{(n_i-m)!}\, n_i^{-l-m} \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x)(k-n_i x)^{l-\nu}$$

$$= \sum_{s=1}^{m} \binom{m}{s} (-1)^{m-s} \sum_{l=1}^{2r-1} \sum_{\nu=0}^{l} \frac{s^{m+\nu}}{(l-\nu)!\,(m+\nu)!}\, f^{(m+l)}(x) \sum_{i=0}^{r-1} C(i,r)$$

$$\sum_{\mu=0}^{m-1} b_\mu \frac{1}{n_i^\mu}\, \frac{1}{n_i^l} \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x) \sum_{j=0}^{l-\nu} c_{j,l-\nu}(x)(k-(n_i-m)x)^j$$

$$= \sum_1 A(s,\mu,l,\nu,j,x)\ f^{(m+l)}(x) \sum_{i=0}^{r-1} C(i,r) \frac{1}{n_i^{l+\mu}}$$

$$\sum_{k=0}^{n_i-m} p_{n_i-m,k}(x)\ (k-(n_i-m)x)^j$$

$$(4.3.10) = \sum_1 A(s,\mu,l,\nu,j,x)\ f^{(m+l)}(x) \sum_{i=0}^{r-1} C(i,r)\frac{1}{n_i^{l+\mu}}\ T_{n_i-m,j}(x)$$

where the sum Σ_1 is over $1 \leq s \leq m$, $0 \leq \mu \leq m-1$, $1 \leq l \leq 2r-1$, $0 \leq \nu \leq l$, $0 \leq j \leq l - \nu$, and

$$A(s,\mu,l,\nu,j,x) = \frac{b_\mu\ s^{m+\nu}\ (-1)^{m-3}\ \binom{m}{s}\ c_{j,l-\nu}(x)}{(l-\nu)!(m+\nu)!}$$

Now , the boundedness of $c_{j,l-\nu}(x), s, \mu, l, \nu$ and j implies that there is a constant K such that

$$(4.3.11) \quad |A(s,\mu,l,\nu,j,x)| \leq K$$

where K is independent of n , f and x .

Further, by (4.1.2a) ,

$$(4.3.12) \sum_1 A(s,\mu,l,\nu,1,x) f^{(m+l)}(x) \sum_{i=0}^{r-1} C(i,r)\frac{1}{n_i^{l+\mu}}\ T_{n_i-m,1}(x) = 0$$

Also for $j = 0$, $T_{n_i-m,0}(x) \equiv 1$ hence the terms corresponding to $j = 0$ in (4.3.10) are

$$A(s,\mu,l,\nu,0,x)\ f^{(m+l)}(x) \sum_{i=0}^{r-1} C(i,r)\ \frac{1}{n_i^{l+\mu}}$$

In the following an estimate for each term corresponding to $j=0$ will be obtained .

Note that, in view of (4.3.11), it is sufficient to estimate

$$S(l,\mu,x) = f^{(m+l)}(x) \sum_{i=0}^{r-1} C(i,r) \frac{1}{n_i^{l+\mu}} \quad , \quad 1 \le l \le 2r-1 \text{ and } 0 \le \mu \le m-1 .$$

First, let $1 \le l \le r$

Clearly $l + \mu \ge 1$. Thus , by (1.5.1d), only those $S(l,\mu,x)$ are nonzero for which $l + \mu \ge r$. So let $l + \mu \ge r$.

Since $1 \le l \le r$, by lemma 4.1.6,

$$|S(l,\mu,x)| \le M(\|f^{(m)}\|_\infty + \|f^{(m+r)}\|_\infty) \sum_{i=0}^{r-1} |C(i,r)| \frac{1}{n_i^{l+\mu}}$$

$$\le M\, n^{-l-\mu} \|f^{(m)}\|_{\infty,2r}$$

$$\le M\, n^{-r} \|f^{(m)}\|_{\infty,2r} \qquad x \in [0,1] \ , \ 1 \le l \le r . \tag{4.3.13}$$

Next, let $r+1 \le l \le 2r-1$

So $l + \mu \ge r$, let $l + \mu = r + k$, $k \in \mathbb{N}_0$. Thus ,

$$|S(l,\mu,x)| \le |f^{(m+l)}(x)| \sum_{i=0}^{r-1} |C(i,r)| \frac{1}{n_i^{l+\mu}}$$

$$\le M\, n^{-r-k} |f^{(m+l)}(x)|$$

$$\le M\, n^{-r-k} |f^{(m+l)}(x)| \left(\frac{2}{A}\, nX \right)^k \qquad \text{(by (4.3.8))}$$

$$\le M \frac{1}{n^r} X^k |f^{(m+l)}(x)| .$$

Since, $l = r+k-\mu$ and $r+1 \le l \le 2r-1$, one gets that $1 \le r-k+\mu \le r-1$, therefore, by lemma 4.1.5,

$$X^k \;|f^{(m+l)}(x)| = X^{\mu}.X^{r-\beta}|f^{(m+2r-\beta)}(x)|$$

$$\leq (\tfrac{1}{4})^{\mu} \; B(\beta)(\|f^{(m)}\|_{\infty} + \|X^r f^{(m+2r)}\|_{\infty})$$

where $\beta = r-k+\mu$.

Thus,

(4.3.14) $\quad |S(l,\mu,x)| \leq M \;\dfrac{1}{n^r}\; \|f^{(m)}\|_{\infty,2r}$

Inequalities (4.3.13-14) and (4.3.11) give that the terms corresponding to $j = 0$ in (4.3.10) are dominated by

$$M\,\frac{1}{n^r}\;\|f\|_{\infty,2r}$$

This together with (4.3.12) takes care of the terms corresponding to $j = 0,1$ in (4.3.10).

Let, $j \geq 2$ this implies that $l \geq 2$.

Therefore, what remains of (4.3.10) to be estimated is

$$\sum\nolimits_2 A(s,\mu,l,\nu,j,x)\;f^{(m+l)}(x)\sum_{i=0}^{r-1} C(i,r)\;\frac{1}{n_i^{\,l+\mu}}\;T_{n_i-m,\,j}(x)$$

where Σ_2 stands for sum over $1 \leq s \leq m$, $0 \leq \mu \leq m-1$, $2 \leq l \leq 2r-1$, $0 \leq \nu \leq l$, $2 \leq j \leq l-\nu$.

Since, $j \geq 2$, using (4.1.3), this can be expressed as

$$\sum\nolimits_2 A(s,\mu,l,\nu,j,x)\;f^{(m+l)}(x)\sum_{i=0}^{r-1} C(i,r)\;\frac{1}{n_i^{\,l+\mu}}$$

$$\sum_{k=1}^{[j/2]} a_{k,j}(x)\;((n_i-m)X)^k$$

$$= \sum\nolimits_2 A(s,\mu,l,\nu,j,x) \sum_{k=1}^{[j/2]} a_{k,j}(x)\ X^k f^{(m+l)}(x)$$

$$\sum_{t=0}^{k} \binom{k}{t} (-m)^{k-t} n_i^t \sum_{i=0}^{r-1} C(i,r) \frac{1}{n_i^{l+\mu}}$$

$$(4.3.15) \quad = \sum\nolimits_3 A(s,\mu,l,\nu,j,k,t,x)\ f^{(m+l)}(x)\ X^k \sum_{i=0}^{r-1} C(i,r) \frac{1}{n_i^{l+\mu-t}}$$

where the sum Σ_3 is over $1 \le s \le m$, $0 \le \mu \le m-1$, $2 \le l \le 2r-1$, $0 \le \nu \le l$, $0 \le j \le l - \nu$, $1 \le k \le [j/2]$, $0 \le t \le k$ and

$$A(s,\mu,l,\nu,j,k,t,x) = A(s,\mu,l,\nu,j,x)\ a_{k,j}(x)\ \binom{k}{t}\ (-m)^{k-t}$$

Since , $a_{k,j}(x)$ is bounded , therefore by (4.3.11),

$$(4.3.17) \quad |A(s,\mu,l,\nu,j,k,t,x)| \le K_1$$

where K_1 is independent of n and f and x.

Thus, it is sufficient to estimate the terms

$$S(l,\mu,k,t,x) = f^{(m+l)}(x)\ X^k \sum_{i=0}^{r-1} C(i,r) \frac{1}{n_i^{l+\mu-t}}$$

where $2 \le l \le 2r-1$, $0 \le \mu \le m-1$, $1 \le k \le [j/2] \le [l/2]$ and $0 \le t \le k$.

Let $2 \le l \le r$. That $l + \mu - t \ge 1$ is obvious . Therefore, in view of (1.5.1d), $S(l,\mu,k,t,x)$ is non-zero only when $l+\mu-t \ge r$.

So, let $l + \mu - t \ge r$.

Thus, using (1.5.1a-b) and that $l+\mu-t\ge r$,

$$|S(l,\mu,k,t,x)| \le X^k |f^{(m+l)}(x)| \sum_{i=0}^{r-1} |C(i,r)| \frac{1}{n_i^{l+\mu-t}}$$

$$\le (\tfrac{1}{4})^k |f^{(m+l)}(x)| \; M \, n^{-r}$$

Since $2 \le l \le r$, lemma 4.1.6 gives.

$$|S(l,\mu,k,t,x)| \le M \frac{1}{n^r} (\|f^{(m)}\|_\infty + \|f^{(m+r)}\|_\infty)$$

(4.3.18) $$\le M \frac{1}{n^r} \|f^{(m)}\|_{\infty,2r} \quad , \qquad x\in[0,1] \; , \; 2 \le l \le r$$

Next, let $r+1 \le l \le 2r-1$.

Again, for the same reason as before, only those $S(l,\mu,k,t,x)$ are non-zero for which $l + \mu - t \ge r$. Let

(4.3.19) $$l + \mu - t = r + q \; , \quad q \in \mathbb{N}_0$$

By (1.5.1a-b),

$$|S(l,\mu k,t,x)| \le X^k \, |f^{(m+l)}(x)| \sum_{i=0}^{r-1} |C(i,r)| \frac{1}{n_i^{l+\mu-t}}$$

$$\le M \frac{1}{n^{l+\mu-t}} X^k \, |f^{(m+l)}(x)|$$

$$\le M \frac{1}{n^{l+\mu-t}} X^k \, |f^{(m+l)}(x)| \left(\frac{2}{A} nX \right)^q \quad , \text{ (by (4.3.8))}$$

$$\le M \; n^{-r} \; X^{k+q} |f^{(m+l)}(x)|$$

But $$X^{k+q} \, |f^{(m+l)}(x)| \le B(r-t-q+\mu)(\|f^{(m)}\|_\infty + \|X^r f^{(m+2r)}\|_\infty) \; .$$

This follows by lemma 4.1.5 together with the fact that $r+1 \le l \le 2r-1$ and (4.3.19) implies $1\le r-t-q+\mu \le r-1$.

Hence ,

(4.3.20) $\quad |S(l,\mu,k,t,x)| \le M \; n^{-r} \; \|f^{(m)}\|_{\infty,2r}.$

Inequalities (4.3.18) and (4.3.20) together with (4.3.17) give that each term in (4.3.16) is majorised by

$$M\; n^{-r}\|f^{(m)}\|_{\infty,2r}.$$

This takes care of the terms corresponding to $j \ge 2$ in (4.3.10). Thus ,

(4.3.21) $\quad |S_{11}(x)| \le M \; n^{-r}\|f^{(m)}\|_{\infty,2r}.$

Remark 4.3.1 : Observe that no restrictions on the range of x were made to obtain the inequalities (4.3.13 and 18) this in conjunction with (4.3.12) allows to conclude that the terms corresponding to $1 \le l \le r$ in (4.3.10) are majorized by

$$M\; n^{-r}\|f^{(m)}\|_{\infty,2r} .$$

The estimate for $S_{12}(x)$ is as follows

$$|S_{12}(x)| \le M \sum_{s=1}^{m} \sum_{i=0}^{r-1} |C(i,r)| \frac{n_i!}{(n_i-m)!} \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x)$$

$$\int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} \left(\frac{k+s}{n_i} - t \right)^{m-1} \left| \int_x^t |t-u|^{2r-1} \; |f^{(m+2r)}(u)|du \right| dt$$

$$\le M \sum_{s=1}^{m} \sum_{i=0}^{r-1} |C(i,r)| \frac{n_i!}{(n_i-m)!} \frac{1}{n_i^{\,m-1}} \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x)$$

$$\cdot \int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} \left| \int_x^t |t-u|^{2r-1} \; |f^{(m+2r)}(u)|du \right| dt$$

$$\le M \sum_{s=1}^{m} \sum_{i=0}^{r-1} n_i \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x) \int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} \left| \int_x^t |t-u|^{2r-1} \ |f^{(m+2r)}(u)| du \right| dt$$

$$\le M \, \|U^r f^{(m+2r)}\|_\infty \sum_{s=1}^{m} \sum_{i=0}^{r-1} n_i \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x) \int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} \left| \int_x^t |t-u|^{2r-1} \ U^{-r} du \right| dt \tag{4.3.22}$$

where $U = u(1-u)$. It will be shown that for $A/n \le x \le 1-A/n$

$$n_i \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x) \int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} \left| \int_x^t |t-u|^{2r-1} \ U^{-r} du \right| dt \le M \, n^{-r} \tag{4.3.23}$$

The terms corresponding to $k = 0$ and $k = n_i-m$ when $s = m$ will be considered separately. The remainder is

$$n_i \sum_{k=1}^{n_i-m} p_{n_i-m,k}(x) \int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} \left| \int_x^t |t-u|^{2r-1} \ U^{-r} du \right| dt$$

where $k < n_i-m$ when $s = m$. It is easy to see that this is dominated by

$$n_i \sum_{k=1}^{n_i-m} p_{n_i-m,k}(x) \int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} (X^{-r} + T^{-r})(t-x)^{2r} dt$$

In view of (1.5.1a), $A/n_i \le A/n \le x \le 1-A/n \le 1-A/n_i$.

Thus, by lemma 4.3.4, the above is $O(1/n_i^r)$ and hence $O(n^{-r})$.

The term corresponding to $k = n_i-m$ when $s = m$ is :

$$B(x) = n_i p_{n_i-m,n_i-m}(x) \int_{\frac{n_i-m}{n_i}}^{1} \left| \int_x^t |t-u|^{2r-1} U^{-r} |du| dt$$

In the following it will be established that

(4.3.24) $\quad B(x) \le M\, n^{-r}$

Due to symmetry similar estimate holds for the term corresponding to $k = 0$ hence establishing (4.3.23). And this together with (4.3.22) will give

(4.3.25) $\quad |S_{12}(x)| \le M\, n^{-r} \, \|f^{(m)}\|_{\infty,2r}$

So we proceed to establish (4.3.24)

$$B(x) = n_i\, x^{n_i-m} \int_{1-\frac{m}{n_i}}^{1} \left| \int_x^t |t-u|^{2r-1} U^{-r} du \right| dt$$

Choosing $A = 1/C$ where C is as in (1.5.1a)

$$B(x) = n_i x^{n_i-m} \int_{1-\frac{m}{n_i}}^{1-\frac{A}{n}} \left| \int_x^t |t-u|^{2r-1} U^{-r} du \right| dt$$

$$+ n_i x^{n_i-m} \int_{1-\frac{A}{n}}^{1} \left| \int_x^t |t-u|^{2r-1} U^{-r} du \right| dt$$

$$= B_1(x) + B_2(x).$$

$A/n \le x \le 1 - A/n$ implies

$$B_2(x) = n_i x^{n_i - m} \int_{1-\frac{A}{n}}^{1} \int_{x}^{t} |t-u|^{2r-1}\, U^{-r} du dt$$

further, $x < 1-A/n \le t \le 1$ therefore $x \le u \le t$ which implies $|t-u| \le 1-u \le 1-x$, consequently ,

$$B_2(x) \le n_i x^{n_i - m} \int_{1-\frac{A}{n}}^{1} \int_{x}^{t} (1-u)^r (1-x)^{r-1} U^{-r} du\ dt,$$

$$\le n_i x^{n_i - m} (1-x)^{r-1} . x^{-r} \int_{1-\frac{A}{n}}^{1} \int_{x}^{t} du\ dt,$$

$$= n_i x^{n_i - m} (1-x)^{2r-1} . X^{-r} \int_{1-\frac{A}{n}}^{1} (t-x) dt$$

$$\le M\ x^{n_i - m} (1-x)^{2r} . X^{-r}$$

Using lemma 4.3.5 and (4.3.8),

(4.3.26) $\qquad B_2(x) \le M\ X^{-r} (n_i - m)^{-2r} \le M\ (nX)^{-r} . n^{-r} \le M\ n^{-r}$

Next , we take up $B_1(x)$.

Here observe that if $\frac{A}{n} < x \le 1 - \frac{m}{n_i}$ then a similar analysis , as done for $B_2(x)$, gives

(4.3.27) $\quad B_1(x) \le M\ n^{-r}$.

So, let $1 - \frac{m}{n_i} \le x \le 1 - \frac{A}{n}$, in this case

$$B_1(x) \le n_i \int_{1-\frac{m}{n_i}}^{1-\frac{A}{n}} \left| \int_{x}^{t} |t-u|^{2r-1}\, U^{-r} du \right| dt$$

$$\le n_i \int_{1-\frac{m}{n_i}}^{1-\frac{A}{n}} |t-x)|^{2r-1} (X^{-r} + T^{-r}) \left| \int_{x}^{t} du \right| dt$$

$$\leq n_i \int_{1-\frac{m}{n_i}}^{1-\frac{A}{n}} |t-x|^{2r} \left(X^{-r} + T^{-r}\right) dt$$

But , $1 - \frac{m}{n_i} \leq x,\ t \leq 1 - \frac{A}{n}$ implies that $|t-x|^{2r} \leq M\, n^{-2r}$ and

$$X^{-r} \leq M\, n^{r},$$

$$T^{-r} \leq M\, n^{r}$$

Thus,

$$B_1(x) \leq M\, n_i n^{-r} \int_{1-\frac{m}{n_i}}^{1-\frac{A}{n}} dt \leq M\, n^{-r} \tag{4.3.28}$$

Inequalities (4.3.26-28) establish (4.3.24) and hence (4.3.25). Now (4.3.22) and (4.3.26) together complete case (a).

Case (b): Let $\min(x,\ 1-x) \leq A/n$.

Again using Taylor's expansion with integral form of remainder, but this time truncating the series after r terms ,

$$S_1(x) = \frac{1}{(m-1)!}\sum_{s=1}^{m} \binom{m}{s} (-1)^{m-s} \sum_{l=1}^{r-1} \frac{1}{l!} f^{(m+l)}(x) \sum_{i=0}^{r-1} C(i,r) \frac{n_i!}{(n_i-m)!}$$

$$\sum_{k=0}^{n_i-m} p_{n_i-m,k}(x) \int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} \left(\frac{k+s}{n_i} - t\right)^{m-1} (t-x)^{l}\, dt$$

$$+ \frac{1}{(m-1)!}\,\frac{1}{(r-1)!} \sum_{s=1}^{m} \binom{m}{s} (-1)^{m-s} \sum_{i=0}^{r-1} C(i,r)\, \frac{n_i!}{(n_i-m)!}$$

$$\sum_{k=0}^{n_i-m} p_{n_i-m,k}(x) \int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} \left(\frac{k+s}{n_i} - t\right)^{m-1} \int_x^t (t-u)^{r-1} f^{(m+r)}(u)\, du\, dt$$

$$= \bar{S}_{11}(x) + \bar{S}_{12}(x)$$

Proceeding along the same lines as in the previous case we get as in (4.3.10).

$$\bar{S}_{11}(x) = \sum\nolimits_1 A(s,u,l,\nu,j,x)\ f^{(m+l)} \sum_{i=0}^{r-1} C(i,r0\ \frac{1}{n_i^{l+\mu}}\ T_{n_i-m,j}(x)$$

where $A(s,\mu,l,\nu,j,x)$ is same as in (4.3.10) except that $1 \le l \le r-1$. Therefore by remark 4.3.1 ,

(4.3.29) $\quad |\bar{S}_{11}(x)| \le M\ n^{-r} \| f^{(m)} \|_{\infty,2r}.$

As for $\bar{S}_{12}(x)$,

$$|\bar{S}_{12}(x)| \le M\ \| f^{(m+r)} \|_{\infty} \sum_{s=1}^{m} \sum_{i=0}^{r-1} \frac{n_i!}{(n_i-m)!} \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x)\ \left(\frac{s}{n_i}\right)^{m-1}$$

$$\int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} \left| \int_x^t |t-u|^{r-1} du \right| dt$$

$$\le M\ \| f^{(m)} \|_{\infty,2r} \sum_{s=1}^{m} \sum_{i=0}^{r-1} \frac{n_i!}{(n_i-m)!}\ \frac{1}{n_i^m}\ n_i \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x)$$

$$\int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} |t-x|^{r-1} \left| \int_x^t du \right| dt$$

$$\le M \|f^{(m)}\|_{\infty,2r} \sum_{s=1}^{m} \sum_{i=0}^{r-1} n_i \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x) \int_{\frac{k}{n_i}}^{\frac{k+s}{n_i}} |t-x|^r dt$$

By lemma 4.3.3,

$$|\bar{S}_{12}(x)| \le M \|f^{(m)}\|_{\infty,2r} \sum_{s=1}^{m} \sum_{i=0}^{r-1} M \frac{1}{n_i^r} \Big(\sum_{\nu=0}^{2r} \sum_{j=0}^{2r-\nu} |T_{n_i-m,j}(x)| \Big)^{1/2}$$

$$\le M \frac{1}{n^r} \|f^{(m)}\|_{\infty,2r} \sum_{s=1}^{m} \sum_{i=0}^{r-1} \Big(\sum_{\nu=0}^{2r} \sum_{j=0}^{2r-\nu} |T_{n_i-m,j}(x)| \Big)^{1/2}$$

Hence to complete the estimate,

(4.3.30) $$|\bar{S}_{12}(x)| \le M \frac{1}{n^r} \|f^{(m)}\|_{\infty,2r} \quad , \min(x, 1-x) \le A/n$$

all that is needed is to show that

$$\sum_{\nu=0}^{2r} \sum_{j=0}^{2r-\nu} |T_{n_i-m,j}(x)| = O(1) \quad , \min(x, 1-x) \le A/n.$$

Now, since $T_{n_i-m,0}(x) \equiv 1$ and $T_{n_i-m,1}(x) \equiv 0$ only terms corresponding to $j \ge 2$ ought to be considered. But $j \ge 2$ implies $0 \le \nu \le 2r-2$. Thus we need to show

(4.3.32) $$\sum_{\nu=0}^{2(r-1)} \sum_{j=2}^{2r-\nu} |T_{n_i-m,j}(x)| = O(1) \quad , \min(x, 1-x) \le A/n .$$

Since, $j \ge 2$ using (4.1.3) the above can be written as

$$\sum_{\nu=0}^{2(r-1)} \sum_{j=2}^{2r-\nu} \left| \sum_{k=1}^{[j/2]} a_{k,j}(x) \, ((n_i-m)X)^k \right|$$

$$\leq \sum_{\nu=0}^{2(r-1)} \sum_{j=2}^{2r-\nu} \sum_{k=1}^{[j/2]} |a_{k,j}(x)| \, ((n_i-m)X)^k$$

$$\leq M \sum_{\nu=0}^{2(r-1)} \sum_{j=2}^{2r-\nu} \sum_{k=1}^{[j/2]} (nX)^k$$

$$\leq \quad M$$

The last inequality follows because $\min(x,1-x) \leq A/n$.

Thus establishing (4.3.31) and hence (4.3.30).

The inequalities (4.3.30) and (4.3.29) together complete the case (b).

Hence we have that for $x \in [0,1]$.

$$|S_1(x)| \leq M \frac{1}{n^r} \|f^{(m)}\|_{\infty,2r}.$$

This together with the estimate for $S_2(x)$ and(4.3.7) establishes the theorem . ■

<u>Corllary</u> <u>4.3.1</u>: Let $f \in A^m_{2r}$, then

$$\|B_{n,r}f-f\|_{m,\infty} \leq M\, n^{-r} \|f\|_{m,\infty,2r} .$$

<u>Proof</u>: Ditzian[25] proved

$$\|B_{n,r}f-f\|_{\infty} \leq M\, n^{-r}(\|f\|_{\infty} + \|X^r f^{(2r)}\|_{\infty}) \qquad , \qquad f \in A_{2r} .$$

This together with lemma 4.1.7 and theorem 4.3.1 establishes the corollary . ■

4.4 INVERSE THEOREM

THEOREM 4.4.1: The following are equivalent

i) $f \in A^{m}_{2r,\phi}$

ii) $\|B_{n,r}f-f\|_{m,\infty} = O(\phi(n^{-1/2}))$

iii) $\|B^{(m)}_{n,r}f-f^{(m)}\|_{\infty} = O(\phi(n^{-1/2}))$

iv) $K(t^{2r}, f^{(m)}) = O(\phi(t))$

where $K(t,f) = \inf_{g \in A^{o}_{2r}} \left\{ \|f-g\|_{\infty} + t\,\|g\|_{\infty,2r} \right\}$

Proof:

i) implies ii)

By (4.1.1), for $f \in C^{m}[0,1]$

$$\|B_n f\|_{m,\infty} = \|B_n f\|_{\infty} + \|B^{(m)}_n f\|_{\infty} \le \|f\|_{\infty} + M\,\|f^{(m)}\|_{\infty} \le M\,\|f\|_{m,\infty}$$

Thus ,

$$\|B_{n,r}f\|_{m,\infty} \le M \max_{0\le i\le r-1} \|B_{n_i}f\|_{m,\infty} \le M\,\|f\|_{m,\infty} .$$

Now suppose $f \in A^{m}_{2r,\phi}$. Therefore for given t there exists a $g_t \in A^{m}_{2r}$ such that

(4.4.1) $\quad \|f-g_t\|_{m,\infty} \le C_1\,\phi(t)$

(4.4.2) $\quad t^{2r}\|g_t\|_{m,\infty,2r} \le C_1\,\phi(t)$

The symbols C_1 , C_2 ... are constants independent of t and n .

Thus ,

$$\|B_{n,r}f-f\|_{m,\infty} \le \|B_{n,r}(f-g_t)\|_{m,\infty} + \|B_{n,r}g_t-g_t\|_{m,\infty} + \|f-g_t\|_{m,\infty}$$

$$\le (M+1)\,\|f-g_t\|_{m,\infty} + \|B_{n,r}g_t-g_t\|_{m,\infty}$$

By corollary 4.3.1 and (4.4.1-2),

$$\|B_{n,r}f-f\|_{m,\infty} \le C_1(M+1)\,\phi(t) + C_1M\,n^{-r}t^{-2r}\phi(t)$$

Taking $t = n^{-1/2}$ (ii) follows .

ii) implies iii) is obvious .

iii) implies iv)

Suppose $\|B_{n,r}^{(m)} f - f^{(m)}\|_\infty = O(\phi(n^{-1/2}))$

To show that $K(t^{2r}, f^{(m)}) = O(\phi(t))$.

$$K(t^{2r}, f^{(m)}) = \inf_{g \in A_{2r}^{o}} \left\{ \|f^{(m)} - g\|_\infty + t^{2r} \|g\|_{\infty,2r} \right\}$$

$$\leq \left\{ \|B_{n,r}^{(m)} f - f^{(m)}\|_\infty + t^{2r} \|B_{n,r}^{(m)} f\|_{\infty,2r} \right\}$$

$$\leq \Big\{ C_3 \, \phi(n^{-1/2})$$

$$+ t^{2r} \left\{ \|B_{n,r}^{(m)}(f-g)\|_{\infty,2r} + \|B_{n,r}^{(m)} g\|_{\infty,2r} \right\} \Big\}$$

where g is such that $g^{(m)} \in A_{2r}^{m}$.

Since $\|B_{n,r} f\|_{\infty,2r} \leq M \max_{0 \leq i \leq r-1} \|B_{n_i} f\|_{\infty,2r}$, by corollaries 4.2.1-2 it follows that

$$K(t^{2r}, f^{(m)}) \leq C_4 \left\{ \phi(n^{-1/2}) + t^{2r} n^{r} K(n^{-r}, f^{(m)}) \right\}$$

Now lemma 1.6.3 gives (iv) .

iv) implies i)

In order to show that $f \in A_{2r,\phi}^{m}$ or in other words that $K(t^{2r}, f) = O(\phi(t))$, it is sufficient to show that for given t there is an g_t in A_{2r}^{m} such that (4.4.1-2) holds .

Note that, for $\phi \in \Phi_{2r}$,

$$(4.4.3) \qquad \frac{t^{2r}}{\phi(t)} \leq \lambda , \qquad 0 < t < c ,$$

where λ is a constant independent of t.

Now suppose $K(t^{2r}, f^{(m)}) = O(\phi(t))$, thus, for given t there is a h_t in A^{o}_{2r} such that

(4.4.4) $\|f^{(m)}-h_t\|_{\infty} \le C_5\ \phi(t)$

(4.4.5) $t^{2r}\ \|h_t\|_{\infty,2r} \le C_5\ \phi(t)$.

Let

$$g_t(x) = \int_0^x \int_0^{s_1} \cdots \int_0^{s_{m-1}} h_t(s)dsds_{m-1}\cdots ds_1$$

$$+ f(0) + xf'(0) + \ldots + \frac{x^{m-1}}{(m-1)!} f^{(m-1)}(0)$$

clearly $g_t^{(m)}(x) = h_t(x)$

Moreover,

$$f(x) = \int_0^x \int_0^{s_1} \cdots \int_0^{s_{m-1}} f^{(m)}(s)dsds_{m-1}\cdots ds_1$$

$$+ f(0) + xf'(0) + \ldots + \frac{x^{m-1}}{(m-1)!} f^{(m-1)}(0)$$

Thus,

$$|f(x)-g_t(x)| \le \int_0^1 |f^{(m)}(s) - h_t(s)|ds \le \|f^{(m)}-h_t\|_{\infty}$$

and $|f^{(m)}(x) - g_t^{(m)}(x)| = |f^{(m)}(x) - h_t(x)| \le \|f^{(m)}-h_t\|_{\infty}$.

Therefore by (4.4.4),

$$\|f-g_t\|_{m,\infty} \le C_5\ \phi(t)$$

Further,

$$|g_t(x)| \le \int_0^1 |h_t(s)|ds + \sum_{i=0}^{m-1} \frac{1}{i!} |f^{(i)}(0)|$$

$$\le \|h_t\|_{\infty} + C_6$$

By (4.4.3 and 4.4.5),

$$\frac{t^{2r}}{\phi(t)} \|g_t\|_\infty \le \frac{t^{2r}}{\phi(t)} \|h_t\|_\infty + C_6 \frac{t^{2r}}{\phi(t)} \le C_7$$

Thus ,

$$t^{2r}\|g_t\|_{m,\infty,2r} = t^{2r}\|g_t\|_\infty + t^{2r}\|g_t^{(m)}\|_{\infty,2r} \le C_8\ \phi(t)$$

This completes the proof of (iv) implies (i) .

Hence the theorem . ■

CHAPTER V

GLOBAL SIMULTANEOUS APPROXIMATION BY LINEAR COMBINATION OF BERNSTEIN-KANTOROVITCH POLYNOMIALS

5.1 PRELIMINARIES

The Bernstein-Kantorovitch polynomials, for functions belonging to $L_p[0,1]$, $1 \leq p \leq \infty$, are given by

$$P_n f(x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt \ , \ n \in \mathbb{N}$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$.

The following lemma gives a relation between $P_n f(x)$ and the Bernstein-polynomials $B_n f(x)$ of the previous chapter.

<u>LEMMA</u> <u>5.1.1</u> Let $f \in L_p[0,1]$, $1 \leq p \leq \infty$. If $F(t) = \int_0^t f(u)du$, then

$$P_n \ f(x) = \frac{d}{dx} B_{n+1} \ F(x)$$

<u>Proof</u>: $\frac{d}{dx} B_{n+1} \ F(x) = (n+1) \sum_{k=0}^{n} \left[F(\frac{k+1}{n+1}) - F(\frac{k}{n+1}) \right] p_{n,k}(x)$

$$= P_n f(x) \ . \ \blacksquare$$

Let $m \in \mathbb{N}_0$, then the above lemma gives

$$(5.1.1)\qquad P_n^{(m)} f(x) = B_{n+1}^{(m+1)} F(x),\ f \in L_p[0,1],\ 1 \le p \le \infty.$$

where F(x) is same as in lemma 5.1.1.

By (5.1.1) and lemma 4.1.1 ,

$$(5.1.2)\qquad P_n^{(m)} f(x) = \frac{(n+1)!}{(n-m)!} \sum_{k=0}^{n-m} p_{n-m,k}(x)\ \Delta_{\frac{1}{n+1}}^{m+1} F\left(\frac{k}{n+1}\right).$$

If in addition $f \in W_p^m[0,1]$, $1 \le p \le \infty$, then as a consequence of mean value theorem ,

$$\Delta_{\frac{1}{n+1}}^{m} F\left(\frac{k}{n+1}\right) = (n+1)^{-m} F^{(m)}\left[\frac{k+m\theta_k}{n+1}\right],\ 0 < \theta_k < 1 .$$

Thus ,

$$P_n^{(m)} f(x) = \frac{(n+1)!}{(n-m)!} (n+1)^{-m} \sum_{k=0}^{n-m} p_{n-m,k}(x)\ \Delta_{\frac{1}{n+1}} F^{(m)}\left[\frac{k+m\theta_k}{n+1}\right]$$

$$(5.1.3)\qquad = \frac{(n+1)!}{(n-m)!} (n+1)^{-m} \sum_{k=0}^{n-m} p_{n-m,k}(x) \int_{\frac{k+m\theta_k}{n+1}}^{\frac{k+m\theta_k+1}{n+1}} f^{(m)}(t)dt .$$

The intermediate Space $W_{p,2r}^{m,\phi}$

Following Winslin[92] , for $1 \le p < \infty$ and $r \in \mathbb{N}$,

$$L_{p,2r} = \Big\{ f \in L_p[0,1]:\ f^{(2r-1)} \in \text{loc. A.C. } (0,1) \text{ and } X^r f^{(2r)} \in L_p[0,1] \Big\}.$$

The spaces $W_{p,2r}^{m}$ are now defined as follows

$$W_{p,2r}^{m} = \left\{ f \in W_p^m[0,1] : f^{(m)} \in L_{p,2r} \right\}$$

These spaces are normed by

$$\|f\|_{m,p,2r} = \|f\|_{m,p} + \|X^r f^{(m+2r)}\|_p$$

where $\|f\|_p = \|f\|_{L_p[0,1]}$ and

$$\|f\|_{m,p} = \|f\|_p + \|f^{(m)}\|_p .$$

The intermediate space $W^{m,\phi}_{p,2r}$ is defined using the Peetre's K-functional

$$K_m(t,f) = \inf_{g \in W^m_{p,2r}} \left\{ \|f-g\|_{m,p} + t\,\|g\|_{m,p,2r} \right\} , \quad f \in W^m_p\,[0,1]$$

For $\phi \in \Phi_{2r}$, let

$$W^{m,\phi}_{p,2r} = \left\{ f \in W^m_p\,[0,1] \; : \; K_m(t^{2r},f) = O(\phi(t)) \right\} .$$

<u>THEOREM</u> <u>5.1.1</u>: The space $W^m_{p,2r}$ is complete with respect to the norm $\|\cdot\|_{m,p,2r}$.

<u>Proof</u>: Let (f_n) be a Cauchy sequence in ($W^m_{p,2r}$, $\|\cdot\|_{m,p,2r}$) then (f_n) and $(X^r f_n^{(m+2r)})$ are cauchy in ($W^m_p[0,1]$, $\|\cdot\|_{m,p}$) and ($L_p[0,1]$, $\|\cdot\|_p$) respectively . But these spaces being complete , there exist f and g in $W^m_p[0,1]$ and $L_p[0,1]$ respectively such that

$\|f_n-f\|_{m,p}$ and $\|X^r f_n^{(m+2r)}-h\|_p$ go to zero as n goes to infinity .

Now define,

$$\tau = X^r D^{m+2r} .$$

Following the notations of definition 1.5.5,

$$D(T_{\tau,p,p}) = \left\{ f : f^{(m+2r-1)} \in \text{loc.A.C.}(0,\infty) \text{ and } \|\tau f\|_p < \infty \right\}$$

Clearly $W^m_{p,2r} = D(T_{\tau,p,p})$

By lemma 1.6.4, $T_{\tau,p,p}$ is closed . Also $T_{\tau,p,p} f_n$ converges to g in the L_p norm . Therefore , using the fact that $T_{\tau,p,p}$ is closed on $D(T_{\tau,p,p})$ one gets

$$f \in D(T_{\tau,p,p}) \quad \text{and} \quad T_{\tau,p,p} f = g$$

From here it follows that $f \in W^m_{p,2r}$ and $g = X^r f^{(m+2r)}$.

Hence the theorem . ∎

<u>LEMMA</u> <u>5.1.2</u>: Let $f \in W^m_p[0,1]$, $1 \le p \le \infty$, then

$$\|P_n f\|_{m,p} \le M \|f\|_{m,p}$$

<u>Proof</u>: From (5.1.3),

$$|P_n^{(m)} f(x)| \le M\,(n+1) \sum_{k=0}^{n-m} p_{n-m,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+m+1}{n+1}} |f^{(m)}(t)|\,dt$$

$$= M \int_0^1 K(n,m,x,t)\, |f^{(m)}(t)|\, dt$$

where $K(n,m,x,t) = (n+1) \sum_{k=0}^{n-m} p_{n-m,k}(x)\, \chi_{n,m,k}(t)$ and $\chi_{n,m,k}(t)$ is the characteristic function of the interval $[\frac{k}{n+1}, \frac{k+m+1}{n+1}]$.

By Holder's inequality,

$$|P_n^{(m)}f(x)| \le M \left[\int_0^1 K(n,m,x,t)dt \right]^{\frac{1}{q}}$$

$$\left[\int_0^1 K(n,m,x,t) \ |f^{(m)}(t)|^p dt \right]^{\frac{1}{p}}$$

Therefore,

$$\int_0^1 |P_n^{(m)}f(x)|^p dx \le M(n+1) \sum_{k=0}^{n-m} \int_0^1 p_{n-m,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+m+1}{n+1}} |f^{(m)}(t)|^p dt \ dx$$

$$\le M \sum_{s=0}^{m} \sum_{k=0}^{n-m} \int_{\frac{k}{n+1}}^{\frac{k+s+1}{n+1}} |f^{(m)}(t)|^p dt$$

$$\le M \sum_{s=0}^{m} \int_0^1 |f^{(m)}(t)|^p dt$$

Thus,

(5.1.4) $$\|P_n^{(m)}f\|_p \le M \|f^{(m)}\|_p$$

Since the same holds for m=0 as well, the lemma follows . ■

<u>LEMMA</u> <u>5.1.3</u>[92]: Let $1 \le p < \infty$ and $f \in L_p[0,1]$ be such that $f^{(2r-1)} \in$ loc. A.C. (0,1) and $X^r f^{(2r)} \in L_p$ [0,1]. Then for $i=0,\ldots,r$, $X^{r-i} f^{(2r-i)} \in L_p[0,1]$ and there exists a constant M such that

$$\|X^{r-i} f^{(2r-i)}\|_p \le M (\|f\|_p + \|X^r f^{(2r)}\|_p).$$

<u>LEMMA</u> <u>5.1.4</u>: Let $1 \le p < \infty$ and $f \in W^m_{p,2r}$, then for $j = 0,\ldots,r$, $f^{(m+j)} \in L_p[0,1]$ and there exists a constant M such that

$$\|f^{(m+j)}\|_p \le M (\|f^{(m)}\|_p + \|X^r f^{(m+2r)}\|_p$$

Proof: Using lemma 5.1.3 for $j = r$ and lemma 1.6.1 for $0 \le j < r$, the lemma follows . ∎

LEMMA 5.1.5: Let $r, m \in \mathbb{N}_o$ and $1 \le p < \infty$ then

$$W^m_{p,2r} \subseteq W^o_{p,2r} \equiv L_{p,2r} \text{ and}$$

$$(5.1.5) \qquad \|f\|_{p,2r} \le M \, \|f\|_{m,p,2r},$$

where $\|f\|_{p,2r} = \|f\|_p + \|X^r f^{(2r)}\|_p$.

Proof: For $m=r=0$ the lemma holds trivially . So let $m, r \ge 1$ and $f \in W^m_{p,2r}$ that is $f^{(m+2r-1)} \in \text{loc A.C.}(0,1)$ and $X^r f^{(m+2r)} \in L_p[0,1]$. That $f^{(2r-1)} \in \text{loc A.C.}(0,1)$ is ofcourse true . In the following we show that $X^r f^{(2r)} \in L_p[0,1]$. This is done in two parts (i) $m \ge r$ and (ii) $1 \le m \le r-1$.

Suppose $m \ge r$, applying lemma 5.1.3,

$$\|f^{(m+r)}\|_p \le M \, \|f^{(m)}\|_{p,2r},$$

moreover, since $m+r \ge 2r$ an application of lemma 1.6.1 together with the above gives us

$$\|f^{(2r)}\|_p \le M \, \|f\|_{m,p,2r} .$$

Thus, using the boundedness of X we get

$$\|X^r f^{(2r)}\|_p \le M \, \|f\|_{m,p,2r},$$

and hence (5.1.5).

Next let $1 \le m < r$.

Using lemma 5.1.3 together with $X \le \frac{1}{4}$,

$$\|X^r f^{(2r)}\|_p \le M \|X^{r-m} f^{(m+2r-m)}\|_p$$
$$\le M \|f\|_{m,p,2r} \quad .$$

Hence the lemma . ∎

5.2 BERNSTEIN TYPE INEQUALITY FOR $f \in W_p^m[0,1]$

<u>Proposition</u> <u>5.2.1</u>: Let $f \in W_p^m[0,1]$, $1 \le p \le \infty$, $m \in \mathbb{N}$ and $r \in \mathbb{N}$. Then

$$\|X^r P_n^{(m+2r)} f\|_p \le M\, n^r \|f^{(m)}\|_p$$

Before we take up the proof of the proposition we mention the following results from [92].

<u>LEMMA</u> <u>5.2.1</u>: Let s be a fixed non-negative integer . Then for $n,k \in \mathbb{N}$ and satisfying $s < k < n-s$

$$T = (n+1) \binom{n}{k} \int_0^1 (x - \frac{k}{n})^{2s} x^{k-s} (1-x)^{n-k-s}\, dx$$
$$= O\,(1/n^s).$$

The capital O-term holding uniformly in k.

<u>LEMMA</u> <u>5.2.2</u>: Let s be a fixed positive integer. Then, for k satisfying $0 \le k \le s$ or $n - s \le k \le n$,

$$(n+1) \binom{n}{k} \int_0^1 (\frac{k}{n} - x)^{2s} x^{k}(1-x)^{n-k}\, dx = O(\frac{1}{n^{2s}})$$

The O-term holding uniformly in k.

<u>Proof</u> (of the proposition 5.2.1): Since $P_n f(x)$ is a polynomial of degree n the result is trivial for $n < m+2r$.

Let n $\geq$ m+2r, the proposition will be proved in two parts .

<u>Case (a):</u> Let min (x, 1-x) $\leq \frac{A}{n-m}$, A is some constant.

By (5.1.2),

$$P_n^{(m+2r)} f(x) = \frac{(n+1)!}{(n-m-2r)!} \sum_{k=0}^{n-m-2r} p_{n-m-2r,k}(x) \Delta^{m+2r+1}_{\frac{1}{n+1}} F(\frac{k}{n+1})$$

where, $F(x) = \int_0^x f(t)\, dt$.

Since $f \in W_p^m[0,1]$, by mean value theorem,

$$\Delta^{m+2r+1}_{\frac{1}{n+1}} F(\frac{k}{n+1}) = \frac{1}{(n+1)^m} \Delta^{2r+1}_{\frac{1}{n+1}} F^{(m)}(\frac{k+m\theta_k}{n+1}),\ 0 < \theta_k < 1$$

$$= \frac{1}{(n+1)^m} \int_{\frac{k+m\theta_k}{n+1}}^{\frac{k+m\theta_k+1}{n+1}} \Delta^{2r}_{\frac{1}{n+1}} f^{(m)}(t)dt.$$

$$= \frac{1}{(n+1)^m} \int_{\frac{k+m\theta_k}{n+1}}^{\frac{k+m\theta_k+1}{n+1}} \sum_{j=0}^{2r} \binom{2r}{j}(-1)^{2r-j} f^{(m)}(t+\frac{j}{n+1})dt.$$

Thus,

$$x^r|P_n^{(m+2r)}f(x)| \leq M\ x^r n^{2r+1} \sum_{j=0}^{2r} \binom{2r}{j} \sum_{k=0}^{n-m-2r} p_{n-m-2r,k}(x)$$

$$\int_{\frac{k+m\theta_k}{n+1}}^{\frac{k+m\theta_k+1}{n+1}} |f^{(m)}(t+\frac{j}{n+1})|\ dt.$$

$$\leq M\, X^r n^{2r} \sum_{j=0}^{2r} (n+1) \sum_{k=0}^{n-m-2r} p_{n-m-2r,k}(x) \int_{\frac{k+j}{n+1}}^{\frac{k+m+j+1}{n+1}} |f^{(m)}(t)|dt$$

(because $0<\theta_k<1$)

$$(5.2.1) \qquad = M\, n^{2r} X^r \int_0^1 G(n,m,r,x,t)\, |f^{(m)}(t)|dt.$$

where,

$$G(n,m,r,x,t) = \sum_{j=0}^{2r} (n+1) \sum_{k=0}^{n-m-2r} p_{n-m-2r,k}(x)\, \chi_{n,m,k,j}(t)$$

and $\chi_{n,m,k,j}(t)$ is the characteristic function of the interval

$\left[\frac{k+j}{n+1}, \frac{k+m+j+1}{n+1} \right]$.

It can be easily verified that

$$\int_0^1 G(n,m,r,x,t)\, dt = (m+1)(2r+1)$$

Now Holder's inequality gives that the right hand side of (5.2.1) does not exceed

$$M\, n^{2r} X^r ((m+1)(2r+1))^{1/q} \left(\int_0^1 G(n,m,r,x,t) |f^{(m)}(t)|^p dt \right)^{1/p}$$

Therefore,

$$|X^r P_n^{(m+2r)} f(x)|^p \leq M\, n^{2rp} X^{rp} \int_0^1 G(n,m,r,x,t) |f^{(m)}(t)|^p dt$$

$$(5.2.2)\qquad \le M\, n^{rp}((n-m)X)^{rp} \sum_{j=0}^{2r} (n+1) \sum_{k=0}^{n-m-2r} p_{n-m-2r,k}(x) \int_{\frac{k+j}{n+1}}^{\frac{k+m+j+1}{n+1}} |f^{(m)}(t)|^p dt.$$

$\min(x,1-x) \le \frac{A}{n-m}$ implies that either $x \le \frac{A}{n-m}$ or $(1-x) \le \frac{A}{n-m}$, in either case

$$(5.2.3)\qquad (n-m)X \le A.$$

Suppose $x \le \frac{A}{n-m}$, then by (5.2.2-3),

$$\int_0^{\frac{A}{n-m}} |X^r P_n^{(m+2r)} f(x)|^p dx \le M\, n^{rp}(n+1) \sum_{j=0}^{2r} \sum_{k=0}^{n-m-2r} \int_0^1 p_{n-m-2r,k}(x)dx \int_{\frac{k+j}{n+1}}^{\frac{k+m+j+1}{n+1}} |f^{(m)}(t)|^p dt.$$

$$\le M\, n^{rp} \sum_{j=0}^{2r} \sum_{k=0}^{n-m-2r} \sum_{s=0}^{m} \int_{\frac{k+s+j}{n+1}}^{\frac{k+s+j+1}{n+1}} |f^{(m)}(t)|^p dt.$$

$$\le M\, n^{rp} \sum_{j=0}^{2r} \sum_{s=0}^{m} \int_0^1 |f^{(m)}(t)|^p dt .$$

Thus,

$$(5.2.4a)\qquad \|X^r P_n^{(m+2r)} f\|_{L_p[0,\frac{A}{n-m}]} \le M\, n^r \|f^{(m)}\|_p.$$

Similarly,

$$(5.2.4b)\qquad \|X^r P_n^{(m+2r)} f\|_{L_p[1-\frac{A}{n-m},1]} \le M\, n^r. \|f^{(m)}\|_p.$$

Case (b): Let $\frac{A}{n-m} \le x \le 1-\frac{A}{n-m}$.

By (5.1.3),

$$P_n^{(m)}f(x) = \frac{(n+1)!}{(n-m)!(n+1)^m} \sum_{k=0}^{n-m} p_{n,m,k}(x) \int_{\frac{k+m\theta_k}{n+1}}^{\frac{k+m\theta_k+1}{n+1}} f^{(m)}(t)dt \;.$$

$$(0<\theta_k<1)$$

Hence,

$$P_n^{(m+2r)}f(x) = \frac{(n+1)!}{(n-m)!(n+1)^m} \sum_{k=0}^{n-m} p_{n-m,k}^{(2r)}(x) \int_{\frac{k+m\theta_k}{n+1}}^{\frac{k+m\theta_k+1}{n+1}} f^{(m)}(t)dt.$$

therefore by lemma 4.1.3 $X^r P_n^{(m+2r)}f(x)$ is a sum of terms of the type

$$S(l,s,x) = X^r \frac{(n+1)!}{(n-m)!(n+1)^m} \frac{1}{X^{2r-l}} (n-m)^l q_{l,s}(x)$$

$$\sum_{k=0}^{n-m} (k-(n-m)x)^{2r-2l-s} p_{n-m,k}(x) \int_{\frac{k+m\theta_k}{n+1}}^{\frac{k+m\theta_k+1}{n+1}} f^{(m)}(t)dt$$

where l, $s \ge 0$, $2r - 2l - s \ge 0$ and $q_{l,s}(x)$ is a polynomial independent of n and k.

Thus,

$$|S(l,s,x)| \le M \frac{(n+1)}{X^{r-l}} (n-m)^l |q_{l,s}(x)| \sum_{k=0}^{n-m} |k-(n-m)x|^{2r-2l-s}$$

$$p_{n-m,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+m+1}{n+1}} |f^{(m)}(t)|dt$$

$$\le M \frac{(n+1)}{X^{r-l}} (n-m)^{l} \sum_{k=0}^{n-m} |k-(n-m)x|^{j} p_{n-m,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+m+1}{n+1}} |f^{(m)}(t)|dt.$$

$$= M \frac{1}{X^{r-l}} (n-m)^{l} \int_0^1 H(n,m,x,t)|f^{(m)}(t)|dt \qquad (j=2r-2l-s)$$

where,

$$H(n,m,x,t) = (n+1) \sum_{k=0}^{n-m} |k-(n-m)x|^{j} p_{n-m,k}(x) \chi_{n,m,k}(t).$$

$\chi_{n,m,k}(t)$ is the characteristic function of the interval $[\frac{k}{n+1},\frac{k+m+1}{n+1}]$.

By Holder's inequality,

$$|S(l,s,x)| \le M \frac{1}{X^{r-l}} (n-m)^{l} \left(\int_0^1 H(n,m,x,t)dt\right)^{1/q}$$

$$\left(\int_0^1 H(n,m,x,t)|f^{(m)}(t)|^{p}dt \right)^{1/p}$$

But,

$$\int_0^1 H(n,m,x,t)dt = (n+1) \sum_{k=0}^{n-m} |k-(n-m)x|^{j} p_{n-m,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+m+1}{n+1}} dt$$

$$= (m+1) \sum_{k=0}^{n-m} |k-(n-m)x|^{j} p_{n-m,k}(x)$$

$$\le (m+1)\left(\sum_{k=0}^{n-m} (k-(n-m)x)^{2j} p_{n-m,k}(x) \right)^{1/2}$$

(by Cauchy-Schwarz inequality)

Since $\frac{A}{n-m} \le x \le 1 - \frac{A}{n-m}$, by corollary 4.1.1,

$$\int_0^1 H(n,m,x,t)\,dt \le M\,((n-m)X)^{j/2}$$

Hence,

$$|S(l,s,x)|^p \le M\,\frac{1}{X^{(r-l)p}}\,(n-m)^{l\cdot p}\,((n-m)X)^{\frac{j}{2}\cdot\frac{p}{q}}$$

$$\int_0^1 H(n,m,x,t)\,|f^{(m)}(t)|^p dt$$

$$= M(n+1)(n-m)^{rp}((n-m)X)^{-j/2}\left[\frac{1}{((n-m)X)^{r-l-j/2}}\right]^p$$

$$\sum_{k=0}^{n-m}|k-(n-m)x|^j\,p_{n-m,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+m+1}{n+1}}|f^{(m)}(t)|^p dt$$

$$\le M\,n^{rp}\left[\frac{1}{(n-m)X}\right]^{p\cdot\frac{s}{2}}((n-m)X)^{-\frac{j}{2}}\left[(n+1)\left(\sum_{k=0}^{j}+\sum_{k=j+1}^{n-m-j-1}+\sum_{k=n-m-j}^{n-m}\right)\right.$$

$$\left.|k-(n-m)X|^j\,p_{n-m,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+m+1}{n+1}}|f^{(m)}(t)|^p dt\right]$$

(5.2.5) $\qquad = S_1(x) + S_2(x) + S_3(x)$, say.

For $\frac{A}{n-m} \le x \le 1 - \frac{A}{n-m}$

(5.2.6) $\qquad \frac{1}{(n-m)X} \le \frac{2}{A}$

Since $s \ge 0$ and $j \ge 0$, by (5.2.6),

$$\int_{\frac{A}{n-m}}^{1-\frac{A}{n-m}} S_1(x)dx \le M \; n^{rp}(n+1) \sum_{k=0}^{j} \int_0^1 |k-(n-m)x|^j \; p_{n-m,k}(x)dx \int_{\frac{k}{n+1}}^{\frac{k+m+1}{n+1}} |f^{(m)}(t)|^p dt$$

$$\le M \; n^{rp}(n-m)^j \sum_{k=0}^{j} \int_0^1 (n-m+1)\left|\frac{k}{n-m} -x\right|^j \; p_{n-m,k}(x)dx \int_{\frac{k}{n+1}}^{\frac{k+m+1}{n+1}} |f^{(m)}(t)|^p dt$$

Now Holder's inequality gives that the right hand side of the above does not exceed

$$M \; n^{rp} \; (n-m)^j \sum_{k=0}^{j} \left[\int_0^1 (n-m+1)\left|\frac{k}{n-m} - x\right|^{2j} p_{n-m,k}(x)dx \right]^{1/2}$$

$$\left[\int_0^1 (n-m+1)p_{n-m,k}(x)dx \right]^{1/2} \int_{\frac{k}{n+1}}^{\frac{k+m+1}{n+1}} |f^{(m)}(t)|^p dt$$

Using lemma 5.2.2 and the fact that $\int_0^1 p_{n,k}(x)dx = \frac{1}{n+1}$

$$\int_{\frac{A}{n-m}}^{1-\frac{A}{n-m}} S_1(x)dx \le M \; n^{rp} \sum_{k=0}^{j} \int_{\frac{k}{n+1}}^{\frac{k+m+1}{n+1}} |f^{(m)}(t)|^p dt. \tag{5.2.5a}$$

along similar lines

$$\int_{\frac{A}{n-m}}^{1-\frac{A}{n-m}} S_3(x)dx \le M \; n^{rp} \sum_{k=n-m-j}^{n-m} \int_{\frac{k}{n+1}}^{\frac{k+m+1}{n+1}} |f^{(m)}(t)|^p dt. \tag{5.2.5b}$$

As for $S_2(x)$ by (5.2.6),

$$\int_{\frac{A}{n-m}}^{1-\frac{A}{n-m}} S_2(x)dx \le Mn^{rp}(n-m)^{-\frac{j}{2}}(n+1)\sum_{k=j+1}^{n-m-j-1}\int_0^1 |k-(n-m)x|^j X^{-\frac{j}{2}} p_{n-m,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+m+1}{n+1}} |f^{(m)}(t)|^p dt\, dx$$

Proceeding exactly on similar lines as for $S_1(x)$, but, using lemma 5.2.1 instead of lemma 5.2.2,

$$\text{(5.2.5c)}\qquad \int_{\frac{A}{n-m}}^{1-\frac{A}{n-m}} S_2(x)dx \le M\, n^{rp}\sum_{k=j+1}^{n-m-j-1}\int_{\frac{k}{n+1}}^{\frac{k+m+1}{n+1}} |f^{(m)}(t)|^p dt.$$

Thus (5.2.5a-c) in conjunction with (5.2.5) give

$$\int_{\frac{A}{n-m}}^{1-\frac{A}{n-m}} |S(l,s,x)|^p dx \le M\, n^{rp}\sum_{k=0}^{n-m}\int_{\frac{k}{n+1}}^{\frac{k+m+1}{n+1}} |f^{(m)}(t)|^p dt.$$

$$= M\, n^{rp}\sum_{l=0}^{m}\sum_{k=0}^{n-m}\int_{\frac{k+l}{n+1}}^{\frac{k+l+1}{n+1}} |f^{(m)}(t)|^p dt.$$

$$\le M\, n^{rp}\sum_{l=0}^{m}\int_{\frac{l}{n+1}}^{\frac{n-m+l+1}{n+1}} |f^{(m)}(t)|^p dt.$$

$$\le M\, n^{rp}\sum_{l=0}^{m}\int_0^1 |f^{(m)}(t)|^p dt.$$

Consequently,

(5.2.7) $\|S(l,s,\cdot)\|_{L_p[\frac{A}{n-m},1-\frac{A}{n-m}]} \le M\, n^r \|f^{(m)}\|_p$.

Since $X^r P_n^{(m+2r)} f(x)$ is a sum of terms of the type $S(l,s,x)$,

$$\|X^r P_n^{(m+2r)} f\|_{L_p[\frac{A}{n-m},1-\frac{A}{n-m}]} \le M\, n^r \|f^{(m)}\|_p$$

This establishes case (b) and hence the proposition . ∎

The following corollary is a consequence of the above proposition and (5.1.4).

<u>Corollary 5.2.1</u>: Let $f \in W_p^m[0,1]$, $1 \le p \le \infty$ and $r \in \mathbb{N}$. Then

$$\|P_n^{(m)} f\|_{p,2r} \le M\, n^r \|f^{(m)}\|_p .$$

5.3 BERNSTEIN TYPE INEQUALITY FOR FUNCTIONS IN $W_{p,2r}^m$

<u>THEOREM 5.3.1</u>: Let $f \in W_{p,2r}^m$, $1 \le p < \infty$. Then,

$$\|X^r P_n^{(m+2r)} f\|_p \le M \;\|f^{(m)}\|_{p,2r} .$$

<u>Proof</u>: By (5.1.2),

$$X^r P_n^{(m+2r)} f(x) = \frac{(n+1)!}{(n-m-2r)!} \sum_{k=0}^{n-m-2r} p_{n-m-2r,k}(x)\; X^r\, \Delta_{\frac{1}{n+1}}^{m+2r+1} F\left(\frac{k}{n+1}\right),$$

where $F(x) = \int_0^x f(t)dt$.

Let,

$$S(n,m,f,x) = \frac{(n+1)!}{(n-m-2r)!} \sum_{k=1}^{n-m-2r-1} p_{n-m-2r,k}(x)\; X^r\, \Delta_{\frac{1}{n+1}}^{m+2r+1} F\left(\frac{k}{n+1}\right),$$

Using (5.1.3) one gets

$$S(n,m,f,x) = \frac{(n+1)!}{(n-m-2r)!}\,\frac{1}{(n+1)^{m+2r}}\,X^r \sum_{k=1}^{n-m-2r-1} p_{n-m-2r,k}(x) \int_{\frac{k+(m+2r)\theta_k}{n+1}}^{\frac{k+(m+2r)\theta_k+1}{n+1}} f^{(m+2r)}(t)\,dt$$

$$(0 < \theta_k < 1)$$

Thus,

$$|S(n,m,f,x)| \le M\,(n+1) \sum_{k=1}^{n-m-2r-1} p_{n-m-2r,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+m+2r+1}{n+1}} |f^{(m+2r)}(t)|\,dt$$

$$\le M\,(n+1) \sum_{k=1}^{n-m-2r-1} p_{n-m-2r,k}(x)\, X^r \frac{1}{(\frac{k}{n+1})^r(1-\frac{(k+m+2r+1)}{n+1})^r} \int_{\frac{k}{n+1}}^{\frac{k+m+2r+1}{n+1}} T^r |f^{(m+2r)}(t)|\,dt$$

$$\le M \left(\frac{m+2r+1}{2r+1}\right)^{2r} (n+1) \sum_{k=1}^{n-m-2r-1} p_{n-m-2r,k}(x)\, X^r \frac{1}{(\frac{k}{n-m+1})^r(\frac{n-m-k-2r}{n-m+1})^r} \int_{\frac{k}{n+1}}^{\frac{k+m+2r+1}{n+1}} T^r |f^{(m+2r)}(t)|\,dt \tag{5.3.1}$$

Next it will be shown that for $1 \le p < \infty$,

$$(5.3.2) \qquad |S(n,m,f,x)|^p \le M\,(n+1) \sum_{k=1}^{n-m-2r-1} p_{n-m-2r,k}(x)$$

$$\frac{X^r}{(\frac{k}{n-m+1})^r(\frac{n-m-k-2r}{n-m+1})^r} \int_{\frac{k}{n+1}}^{\frac{k+m+2r+1}{n+1}} \left[T^r|f^{(m+2r)}(t)|\right]^p dt$$

Observe that for p = 1, (5.3.1) is exactly the above inequality except possibly for the multiplicative constant. By Holder's inequality,

$$|S(n,m,f,x)| \le M \left[\sum_{k=1}^{n-m-2r} p_{n-m-2r,k}(x) \frac{X^r}{(\frac{k}{n-m+1})^r(\frac{n-m-k-2r}{n-m+1})^r} \right]^{1/q}$$

$$\left[\sum_{k=1}^{n-m-2r} p_{n-m-2r,k}(x) \frac{X^r}{(\frac{k}{n-m+1})^r(\frac{n-m-k-2r}{n-m+1})^r}\right.$$

$$\left.\left[(n+1) \int_{\frac{k}{n+1}}^{\frac{k+m+2r+1}{n+1}} T^r|f^{(m+2r)}(t)|dt\right]^p\right]^{1/p}$$

Winslin[92] has shown that

$$\sum_{k=1}^{n-2r-1} p_{n-2r,k}(x) \frac{X^r}{[(\frac{k}{n+1})(\frac{n-k-2r}{n+1})]^r} \le C$$

where C is a constant independent of n.

So, in view of this and Jensen's inequality,

$$|S(n,m,f,x)| \le MC^{1/q}\left[(n+1)\sum_{k=1}^{n-m-2r} p_{n-m-2r,k}(x)\frac{X^r}{(\frac{k}{n-m+1})^r(\frac{n-m-k-2r}{n-m+1})^r}\int_{\frac{k}{n+1}}^{\frac{k+m+2r+1}{n+1}}\left[T^r|f^{(m+2r)}(t)|\right]^p dt\right]^{1/p}$$

From here (5.3.2) follows immediately.

To complete the estimate for S(n,m,f,x) following is required,

$$(5.3.3)\qquad (n+1)\frac{1}{(\frac{k}{n+1})^r(\frac{n-k-2r}{n+1})^r}\int_0^1 X^r p_{n-2r,k}(x)dx \le C,\quad 1\le k\le n-2r-1$$

where C is a constant independent of n and k.

This has been established in [92,pp155] .

Using (5.3.3), with n replaced by n-m, in (5.3.2),

$$\int_0^1 |S(n,m,f,x)|^p\,dx \le M\,C.\sum_{k=1}^{n-m-2r-1}\int_{\frac{k}{n+1}}^{\frac{k+m+2r+1}{n+1}}\left[T^r|f^{(m+2r)}(t)|\right]^p dt$$

$$\le M\sum_{k=1}^{n-m-2r-1}\sum_{j=0}^{2r+m}\int_{\frac{k+j}{n+1}}^{\frac{k+j+1}{n+1}}\left[T^r|f^{(m+2r)}(t)|\right]^p dt$$

$$\le M\sum_{j=0}^{2r+m}\int_{\frac{1+j}{n+1}}^{\frac{n-m-2r+j}{n+1}}\left[T^r|f^{(m+2r)}(t)|\right]^p dt$$

$$\le M\sum_{j=0}^{2r+m}\int_0^1(T^r|f^{(m+2r)}(t)|)^p\,dt$$

(5.3.4) $$\leq M \quad \|X^r f^{(m+2r)}\|_p^p$$

The terms $S_o(n,m,f,x)$ and $S_{n-m-2r}(n,m,f,x)$ corresponding to $k=0$ and $k = n-m-2r$ respectively, will be taken up now.

(5.3.5) $$S_o(n,m,f,x) = \frac{(n+1)!}{(n-m-2r)!} X^r \; p_{n-m-2r,0}(x) \; \Delta^{2r+m+1}_{\frac{1}{n+1}} F(0)$$

Now,

$$\Delta^{r+m+1}_{\frac{1}{n+1}} F(x) = \int_0^{\frac{1}{n+1}} \cdots \int_0^{\frac{1}{n+1}} f^{(m+r)} \; (x+t_1+ \; \ldots \; + t_{m+r})dt_1 \ldots dt_{m+r+1}$$

for $0 \leq x \leq \frac{m+r}{n+1}$.

Thus, by Jensen's inequality,

$$|\Delta^{m+r+1}_{\frac{1}{n+1}} F(x)|^p \leq \frac{1}{(n+1)^{(m+r+1)(p-1)}}$$

$$\int_0^{\frac{1}{n+1}} \cdots \int_0^{\frac{1}{n+1}} |f^{(m+r)} \; (x+t_1+ \; \ldots \; + t_{m+r})|^p dt_1 \ldots dt_{m+r+1}$$

$(1<p<\infty)$

$$= \frac{1}{(n+1)^{(m+r+1)(p-1)}} \int_0^{\frac{1}{n+1}} \cdots \int_0^{\frac{1}{n+1}}$$

$$\int_{x+t_2+\ldots+t_{m+r+1}}^{x+t_2+\ldots+t_{m+r+1}+\frac{1}{m+1}} |f^{(m+r)}(s)|^p ds dt_2 \ldots dt_{m+r+1}$$

$$\leq \frac{1}{(n+1)^{(m+r+1)(p-1)}} \; \frac{1}{(n+1)^{m+r}} \; \|f^{(m+r)}\|^p_{L_p[0,1]}$$

Consequently,

(5.3.5a) $$|\Delta^{m+2r+1}_{\frac{1}{n+1}} F(0)| = |\Delta^{r}_{\frac{1}{n+1}} \; \Delta^{m+r+1}_{\frac{1}{n+1}} F(0)|$$

$$\leq \sum_{j=0}^{m+r} \binom{m+r}{j} \left| \Delta_{\frac{1}{n+1}}^{m+r+1} F\left(\frac{j}{n+1}\right) \right|$$

$$(5.3.6) \qquad \leq M \frac{1}{(n+1)^{m+r+1-1/p}} \| f^{(m+r)} \|_p$$

Further using Sterling's formula

$$\| X^r p_{n-m-2r,0}(x) \|_{L_p[0,1]}^p = \int_0^1 x^{(n-m)p-rp} (1-x)^{rp}\, dx$$

$$= \frac{\Gamma(\,(n-m)p - rp + 1)\,\Gamma(rp+1)}{\Gamma(\,(n-m)p + 2)}$$

$$= \frac{((n-m)p-rp+1)^{(n-m)p-rp+1/2} e^{-((n-m)p-rp+1)} (2\pi)^{\frac{1}{2}} e^{\frac{\theta_1}{12((n-m)p-rp+1)}}\, \Gamma(rp+1)}{((n-m)p+2)^{(n-m)p+3/2} e^{-((n-m)p+2)} (2\pi)^{1/2}\, e^{\frac{\theta_2}{12((n-m)p+2)}}}$$

where $0 < \theta_1 < \theta_2 < 1$.

Hence,

$$(5.3.7) \qquad \| X^r p_{n-m-2r,0}(x) \|_p \leq M\, (n+1)^{-r-\frac{1}{p}} \qquad (1 \leq p < \infty) .$$

The inequalities (5.3.6-7) together with (5.3.5) give

$$\| S_o(n,m,f,\cdot) \|_p \leq M \| f^{(m+r)} \|_p \qquad (1<p<\infty)$$

Due to symmetry, in a similar fashion

$$\| S_{n-m-2r}(n,m,f,\cdot) \|_p \leq M \| f^{(m+r)} \|_p \qquad (1<p<\infty)$$

The estimates for $S_o(n,m,f,x)$ and $S_{n-m-2r}(n,m,f,x)$ together with (5.3.4) complete the proof of the theorem 5.3.1 for $1 < p < \infty$.

For $p = 1$, let

$$\Delta_{\frac{1}{n+1}}^{r+m+1} F(x) = \int_0^{\frac{1}{n+1}} \cdots \int_0^{\frac{1}{n+1}} f^{(m+r)}(x+t_1+ \ldots + t_{m+r})dt_1 \ldots dt_{m+r+1},$$

$$0 \le x \le \frac{m+r}{n+1} .$$

Thus,

$$|\Delta_{\frac{1}{n+1}}^{r+m+1} F(x)| \le \int_0^{\frac{1}{n+1}} \cdots \int_0^{\frac{1}{n+1}} \int_{x+t_2+\ldots+t_{m+r+1}}^{x+t_2+\ldots+t_{m+r+1}+\frac{1}{m+1}} |f^{(m+r)}(s)|\, ds dt_2 \ldots dt_{m+r+1}$$

$$\le (n+1)^{-m-r} \|f^{(m+r)}\|_1 .$$

Consequently, by (5.3.5a),

$$|\Delta_{\frac{1}{n+1}}^{2r+m+1} F(0)| \le (n+1)^{-m-r} \|f^{(m+r)}\|_1 .$$

Now in view of (5.3.7) the above inequality along with (5.3.5) gives

$$\|S_o(n,m,f,\cdot)\|_1 \le M \|f^{(m+r)}\|_1 .$$

Further due to symmetry

$$\|S_{n-m-2r}(n,m,f,\cdot)\|_1 \le M \|f^{(m+r)}\|_1 .$$

These together with (5.3.4) proves the theorem for $p = 1$ as well. ■

<u>Corollary 5.3.1</u>: Let $f \in W^m_{p,2r}$, $1 \le p < \infty$, then

$$\|P_n^{(m)} f\|_{p,2r} \le M \|f^{(m)}\|_{p,2r} .$$

<u>Proof</u>: Follows from the above theorem and (5.1.4). ■

5.4 ORDER OF APPROXIMATION FOR $f \in W^{m}_{p,2r}$, $1 < p < \infty$

In this section a direct theorem for simultaneous approximation by linear combination of Bernstein-Kantorovitch polynomial is established.

<u>THEOREM</u> <u>5.4.1</u>: Let $f \in W^{m}_{p,2r}$, $1 < p < \infty$. Then

$$\|P^{(m)}_{n,r}f - f^{(m)}\|_p \le M\, n^{-r}\, \|f^{(m)}\|_{p,2r}$$

<u>Proof</u>: Let $S(x) = P^{(m)}_{n,r}f(x) - f^{(m)}(x)$

$$= \sum_{i=0}^{r-1} C(i,r)\left[\, P^{(m)}_{n_i}f(x) - f^{(m)}(x)\right]$$

$$= \sum_{i=0}^{r-1} C(i,r)\left[\frac{(n_i+1)!}{(n_i-m)!}\sum_{k=0}^{n_i-m} p_{n_i-m,k}(x).\Delta^{m+1}_{\frac{1}{n_i+1}} F\left(\frac{k}{n_i+1}\right) - f^{(m)}(x)\right]$$

where $F(x) = \int_0^x f(t)dt$.

Use of (5.1.2) has been made to get the above.

By lemma 3.2.3,

$$S(x) = \sum_{i=0}^{r-1} C(i,r)\Bigg[\frac{(n_i+1)!}{(n_i-m)!}\,\frac{1}{m!}\sum_{s=0}^{m+1}\binom{m+1}{s}(-1)^{m+1-s}$$

$$\sum_{k=0}^{n_i-m} p_{n_i-m,k}(x)\int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}}\left(\frac{k+s}{n_i+1} - t\right)^m f^{(m)}(t)dt - f^{(m)}(x)\Bigg]$$

By (1.5.1d),

$$S_2(x) = f^{(m)}(x) \sum_{j=1}^{m} b_j \sum_{k=r}^{\infty} a_k \sum_{i=0}^{r-1} C(i,r)\; n_i^{-k}$$

Thus,

$$|S_2(x)| \le |f^{(m)}(x)| \sum_{j=1}^{m} |b_j| \sum_{k=r}^{\infty} |a_k| . \sum_{i=0}^{r-1} |C(i,r)|\; n_i^{-k}$$

$$\le M\, n^{-r} . |f^{(m)}(x)| \sum_{j=1}^{m} \sum_{k=0}^{\infty} \sum_{i=0}^{r-1} n^{-k} \qquad \text{(using (1.5.1a-b))}$$

$$\le M\, n^{-r} . |f^{(m)}(x)|$$

Therefore,

$$\text{(5.4.3)} \quad \|S_2\|_{L_p[0,1]} \le M\, n^{-r}\, \|f^{(m)}\|_{L_p[0,1]} \le M\, n^{-r}\, \|f^{(m)}\|_{p,2r}$$

Remark 5.4.1 : Inequality (5.4.3) holds good for $1 \le p < \infty$.

The analysis for $S_1(x)$ will be carried out in two parts.

Case i: Let $\frac{A}{n+1} \le x \le 1 - \frac{A}{n+1}$, where A is some constant.

$$S_1(x) = \sum_{i=0}^{r-1} C(i,r) \frac{(n_i+1)!}{(n_i-m)!} \frac{1}{m!} \sum_{s=0}^{m+1} \binom{m+1}{s} (-1)^{m+1-s} \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x)$$

$$\int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} \left(\frac{k+s}{n_i+1} - t \right)^m (f^{(m)}(t) - f^{(m)}(x))dt$$

Taylor's expansion with integral form of remainder gives

$$f^{(m)}(t) - f^{(m)}(x) = \sum_{l=1}^{2r-1} \frac{1}{l!}(t-x)^l \, f^{(m+l)}(x) + \frac{1}{(2r-1)!}\int_x^t (t-u)^{2r-1} f^{(m+2r)}(u)du.$$

Thus,

$$S_1(x) = \frac{1}{m!}\sum_{s=0}^{m+1} \begin{bmatrix} m+1 \\ s \end{bmatrix} (-1)^{m+1-s} \sum_{l=1}^{2r-1} \frac{1}{l!} f^{(m+l)}(x) \sum_{i=0}^{r-1} C(i,r)$$

$$\frac{(n_i+1)!}{(n_i-m)!}\sum_{k=0}^{n_i-m} p_{n_i-m,k}(x) \int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} \left[\frac{k+s}{n_i+1} - t\right]^m (t-x)^l dt$$

$$+ \frac{1}{(2r-1)!}\frac{1}{m!}\sum_{s=0}^{m+1} \begin{bmatrix} m+1 \\ s \end{bmatrix} (-1)^{m+1-s} \sum_{i=0}^{r-1} C(i,r) \frac{(n_i+1)!}{(n_i-m)!}.$$

$$\cdot\sum_{k=0}^{n_i-m} p_{n_i-m,k}(x) \int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} \left[\frac{k+s}{n_i+1} - t\right]^m \int_x^t (t-u)^{2r-1} f^{(m+2r)}(u)dudt$$

$$(5.4.4) \qquad = S_{11}(x) + S_{12}(x)$$

Applying lemma 3.2.1 with $a = k/(n_i+1)$ and $h = s/(n_i+1)$,

$$S_{11}(x) = \frac{1}{m!}\sum_{s=0}^{m+1} \begin{bmatrix} m+1 \\ s \end{bmatrix} (-1)^{m+1-s} \sum_{l=1}^{2r-1} \frac{1}{l!} f^{(m+l)}(x) \sum_{\nu=0}^{l} \frac{m!\,.\,l!\,.\,s^{m+\nu+1}}{(m+\nu+1)!(l-\nu)!}$$

$$\times \sum_{i=0}^{r-1} C(i,r) \frac{(n_i+1)!}{(n_i-m)!} (n_i+1)^{-l-m-1} \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x).(k-(n_i+1)x)^{l-\nu}$$

Using lemmas 4.3.1 and 4.3.2, this can be shown to be equal to

$$\sum_{s=0}^{m+1}\binom{m+1}{s}(-1)^{m+1-s}\sum_{l=1}^{2r-1}f^{(m+l)}(x)\sum_{\nu=0}^{l}\frac{m!.l!.s^{m+\nu+1}}{(m+\nu+1)!(l-\nu)!}\sum_{i=0}^{r-1}C(i,r)$$

$$\sum_{\mu=0}^{m}b_\mu\ (n_i+1)^{-l-\mu}\sum_{j=0}^{l-\nu}c_{j,l-\nu}(x)\sum_{k=0}^{n_i-m}p_{n_i-m,k}(x).(k-(n_i-m)x)^j$$

In a more compact form, one has

(5.4.5) $\quad S_{11}(x) = \Sigma_1 A(s,l,\nu,\mu,j;x).S(n,m,l,\mu,j;f,x)$

where Σ_1 represents a sum over $0 \le s \le m+1$, $1 \le l \le 2r-1$, $0 \le \mu \le m$, $0 \le \nu \le l$ and $0 \le j \le l-\nu$,

$$A(s,l,\nu,\mu,j;x) = \binom{m+1}{s}(-1)^{m+1-s}\frac{1}{(m+\nu+1)!(l-\nu)!}s^{m+\nu+1}b_\mu c_{j,l-\nu}(x)$$

and $S(n,m,l,\mu,j;f,x) = f^{(m+l)}(x)\sum_{i=0}^{r-1}C(i,r).(n_i+1)^{-l-\mu}\ T_{n_i-m,j}(x)$

where $T_{n_i-m,j}(x) = \sum_{k=0}^{n_i-m}p_{n_i-m,k}(x).(k-(n_i-m)x)^j$

In what follows it will be shown that for $1 \le l \le 2r-1$, $0 \le \mu \le m$ and $0 \le j \le l-\nu$ where $0 \le \nu \le l$.

(5.4.6) $$\|S(n,m,l,\mu,j;f,x)\|_{L_p[\frac{A}{n+1},1-\frac{A}{n+1}]} \le M\ n^{-r}\ \|f^{(m)}\|_{p,2r}\ ,\quad (1\le p<\infty)$$

Before this is taken up observe that

(5.4.7) $\quad |A(s,l,\nu,\mu,j;x)| \le K$

where K is a constant independent of n and x.

As for establishing (5.4.6), by (4.1.2a),

(5.4.8) $\quad S(n,m,l,\mu,1;f,x) \equiv 0, \qquad 1 \le l \le 2r-1$ and $0 \le \mu < m+1$

and

$$S(n,m,l,\mu,0;f,x) = f^{(m+l)}(x) \sum_{i=0}^{r-1} C(i,r).(n_i+1)^{-l-\mu}$$

$$= f^{(m+l)}(x) \sum_{i=0}^{r-1} C(i,r) \sum_{q=l+\mu}^{\infty} \beta.n_i^{\,q}$$

(by (5.4.3))

$$= f^{(m+l)}(x) \quad \sum_{q=r}^{\infty} \beta.\sum_{i=0}^{r-1} C(i,r)\ n_i^{\,q}$$

(using (1.5.1d)

Thus,

$$\|S(n,m,l,\mu,0;f,\cdot)\|_p \le \|f^{(m+l)}\|_p \sum_{q=r}^{\infty} |\beta_q| \sum_{i=0}^{r-1} |C(i,r)|n_i^{\,q}$$

$$\le M\ n^{-r}\|f^{(m+l)}\|_p$$

Now by lemma 5.1.4,

(5.4.9) $\|S(n,m,l,\mu,0;f,\cdot)\|_p \le M\ n^{-r}\|f^{(m)}\|_{p,2r}\ ,\ 1 \le l \le r.$

Next let $r+1 \le l \le 2r-1$,

$$|S(n,m,l,\mu,0;f,x)| \le |f^{(m+l)}(x)| \sum_{i=0}^{r-1} |C(i,r)|(n_i+1)^{-l-\mu}$$

$$\le M\ (n+1)^{-l-\mu}\ |f^{(m+l)}(x)|.$$

Now, $r+1 \le l \le 2r-1$ and $0 \le \mu \le m$ imply $l+\mu > r$. So let $l+\mu=r+\alpha$, $\alpha \in \mathbb{N}$. Further $\frac{A}{n+1} \le x \le 1-\frac{A}{n+1}$ implies $\frac{2}{A}(n+1)X \ge 1$.

Consequently,

$$|S(n,m,l,\mu,0;f,x)| \le M\ (n+1)^{-r-\alpha}\ |f^{(m+l)}(x)|$$

Now since $\frac{A}{n+1} \le x \le 1-\frac{A}{n+1}$,

$$|S(n,m,l,\mu,j;f,x)| \le M \sum_{k=1}^{[j/2]} \sum_{q=0}^{k} (n+1)^{-l-\mu+q}\ X^k |f^{(m+l)}(x)| \left(\frac{2}{A}(n+1)X\right)^{\alpha}$$

where $\alpha \in \mathbb{N}_o$ is such that $l+\mu-q = r+\alpha$, let $\beta = r-\alpha+\mu-q$, then

$$|S(n,m,l,\mu,j;f,x)| \le M\ n^{-r} \sum_{k=1}^{[j/2]} \sum_{q=0}^{k} X^{k+\alpha-r+\beta}\ X^{r-\beta} |f^{(m+2r-\beta)}(x)|$$

$$\le M\ n^{-r} \sum_{k=1}^{[j/2]} \sum_{q=0}^{k} X^{r-\beta} |f^{(m+2r-\beta)}(x)| \qquad \text{(because } k+\alpha+\beta-r \ge 0\text{)}$$

$l+\mu-q = r+\alpha$, $r+1 \le l \le 2r-1$ implies $1 \le \beta \le r-1$, thus, applying lemma 5.1.3,

$$\|S(n,m,l,\mu,j;f,x)\|_{L_p[\frac{A}{n+1},1-\frac{A}{n+1}]} \le M\ n^{-r}\ \|f^{(m)}\|_{p,2r}\ , \quad 1 \le p < \infty$$

This together with (5.4.8) and the inequalities (5.4.?-10&13-14) establishes (5.4.6) which in turn, along with (5.4.7) gives

$$(5.4.15) \quad \|S_{11}\|_{L\ [\frac{A}{n+1}\ ,1-\frac{A}{n+1}]} \le M\ n^{-r}\ \|f^{(m)}\|_{p,2r}\ , \quad 1 \le p < \infty.$$

Remark 5.4.2: Following (5.4.8-9&13) we have that (5.4.6) holds irrespective of the choice of x for $1 \le l \le r$.

Remark 5.4.3: The estimate (5.4.15) for $S_{11}(x)$ is good for $1 \le p < \infty$.

The remainder term $S_{12}(x)$ will be taken up now.

$$|S_{12}(x)| \le M \sum_{s=0}^{m+1} \sum_{i=0}^{r-1} \frac{(n_i+1)!}{(n_i-m)!}\left(\frac{s}{n_i+1}\right)^m$$

$$\sum_{k=0}^{n_i-m} p_{n_i-m,k}(x) \int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} \left|\int_x^t |t-u|^{2r-1}|f^{(m+2r)}(u)|du\right|dt$$

$$\le M \sum_{s=0}^{m+1} \sum_{i=0}^{r-1} (n_i+1) \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x) \int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} \left|\int_x^t |t-u|^{2r-1}|f^{(m+2r)}(u)|du\right|dt$$

(5.4.16) $$= M \sum_{s=0}^{m+1} \sum_{i=0}^{r-1} S(s,i,x).$$

It is intended to establish that for $0 \le s \le m+1$ and $0 \le i \le r-1$

(5.4.17) $$\|S(s,i,\cdot)\|_{L_p[\frac{A}{n+1},1-\frac{A}{n+1}]} \le M n^{-r} \|f^{(m)}\|_{p,2r}$$

and thus,

(5.4.18) $$\|S_{12}\|_{L_p[\frac{A}{n+1},1-\frac{A}{n+1}]} \le M n^{-r} \|f^{(m)}\|_{p,2r}.$$

Observe that (5.4.17) holds trivially for $s = 0$, therefore henceforward $s \ge 1$.

Let

$$S_r(s,i,x) = (n_i+1) \sum_{k=1}^{n_i-m} p_{n_i-m,k}(x) \int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} \left|\int_x^t |t-u|^{2r-1}|f^{(m+2r)}(u)|du\right|dt$$

where $k \le n_i-m-1$ when $s = m+1$.

Further,

$$S_r(s,i,x) \le (n_i+1)\sum_{k=1}^{n_i-m} p_{n_i-m,k}(x)\int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} |t-x|^{2r-1}\left|\int_x^t \frac{U^r|f^{(m+2r)}(u)|}{U^r}du\right|dt$$

$$\le (n_i+1)\sum_{k=1}^{n_i-m} p_{n_i-m,k}(x)\int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} (t-x)^{2r}(X^{-r}+T^{-r})$$

$$\frac{1}{|t-x|}\left|\int_x^t U^r|f^{(m+2r)}(u)|du\right|dt$$

$$\le G(x).(n_i+1)\sum_{k=1}^{n_i-m} p_{n_i-m,k}(x)\int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} (t-x)^{2r}(X^{-r}+T^{-r})dt$$

where $U = u(1-u)$, $T = t(1-t)$ and $G(x)$ is the Hardy-Littlewood majorant of $X^r f^{(m+2r)}(x)$.

Now, $\frac{A}{n+1} \le x \le 1-\frac{A}{n+1}$ and $n \le n_i \le C.n$ imply $\frac{A}{n_i+1} \le x \le 1-\frac{A}{n_i+1}$

also $k < n_i-m$ for $s = m+1$.

Thus, by lemma 4.3.4, with $\alpha=2r$, $\beta=r$ and n,m replaced n_i+1, $m+1$ respectively,

$$S_r(s,i,x) \le M\,(n_i+1)^{-r}\,G(x)$$
$$\le M\,n^{-r}\,G(x).$$

Now lemma 1.6.2 gives

$$\|S_r(s,i,\cdot)\|_{L_p[\frac{A}{n+1},1-\frac{A}{n+1}]} \le M\,n^{-r}\|X^r f^{(m+2r)}\|_{L_p[\frac{A}{n+1},1-\frac{A}{n+1}]},\quad 1<p<\infty$$

(5.4.19) $$\le M\,n^{-r}\,\|f^{(m)}\|_{p,2r}.$$

What remains, to be estimated in order to complete the analysis

for $S_{12}(x)$, is the term corresponding to (i) k=0 and (ii) $k=n_i-m$ when $s = m+1$.

The term corresponding to $k = 0$ is

$$S_o(x) = (n_i+1)\ p_{n_i-m,0}(x)\int_0^{\frac{s}{n_i+1}} \left|\int_x^t \frac{|t-u|^{2r-1}U^r|f^{(m+2r)}(u)|}{U^r}du\right|dt$$

Choosing $A = \frac{1}{C}$, where C is same as in (1.5.1a),

$$S_o(x) = (n_i+1)\ (1-x)^{n_i-m}\int_0^{\frac{A}{n+1}} \left|\int_x^t \frac{|t-u|^{2r-1}U^r|f^{(m+2r)}(u)|}{U^r}du\right|dt$$

$$+ (n_i+1)\ (1-x)^{n_i-m}\int_{\frac{A}{n+1}}^{\frac{s}{n_i+1}} \left|\int_x^t \frac{|t-u|^{2r-1}U^r|f^{(m+2r)}(u)|}{U^r}du\right|dt$$

$$= A_1(x) + A_2(x)$$

For, $A_1(x)$ $0 \le t \le \frac{A}{n+1} \le x \le 1-\frac{A}{n+1}$ which implies $|t-u| \le u \le x$ and $\frac{1}{1-u} \le \frac{1}{1-x}$.

Thus,

$$A_1(x) \le (n_i+1)\ (1-x)^{n_i-m}\int_0^{\frac{A}{n+1}}\int_t^x \frac{|t-u|^{r-1}U^r|f^{(m+2r)}(u)|}{(1-u)^r}dudt$$

$$\le (n_i+1)\ (1-x)^{n_i-m-r}\int_0^{\frac{A}{n+1}}(x-t)^r\frac{1}{x-t}\int_t^x U^r|f^{(m+2r)}(u)|dudt$$

$$\le (n_i+1)\ (1-x)^{n_i-m-r}\ x^r.G(x)\int_0^{\frac{A}{n+1}} dt$$

$$\le M\ G(x)\ X^{-r}\ (1-x)^{n_i-m}\ x^{2r}.$$

$G(x)$ is the Hardy-Littlewood majorant of $U^r f^{(m+2r)}$. Now,

applying lemma 4.3.5 and using (1.5.1a) the above expression can be shown not to exceed

$$M\ G(x)\ X^{-r}(n+1)^{-2r}$$

$$\le M\ G(x)\ (n+1)^{-r} \qquad \left(\text{because } \frac{A}{n+1} \le x \le 1-\frac{A}{n+1}\right).$$

Consequently $A_1(x) \le M\ n^{-r}G(x)$. Therefore,

$$\|A_1\|_{L_p[\frac{A}{n+1},1-\frac{A}{n+1}]} \le M\ n^{-r}\|G\|_{L_p[\frac{A}{n+1},1-\frac{A}{n+1}]} \le M\ n^{-r}\|G\|_p.$$

Now, by lemma 1.6.2,

$$\|A_1\|_{L_p[\frac{A}{n+1},1-\frac{A}{n+1}]} \le M\ n^{-r}\|X^r f^{(m+2r)}\|_p \qquad (1<p<\infty)$$

(5.4.20) $$\le M\ n^{-r}\|f^{(m)}\|_{p,2r}.$$

Now we take up $A_2(x)$. Recall that

$$A_2(x) = (n_i+1)\ (1-x)^{n_i-m} \int_{\frac{A}{n+1}}^{\frac{s}{n_i+1}} \left|\int_x^t \frac{|t-u|^{2r-1}U^r|f^{(m+2r)}(u)|}{U^r}\ du\right| dt$$

here $\frac{A}{n+1} \le t \le \frac{s}{n_i+1}$ also $\frac{A}{n+1} \le x \le 1-\frac{A}{n+1}$. For $\frac{s}{n_i+1} \le x \le 1-\frac{A}{n+1}$, proceeding in the same fashion as for $A_1(x)$,

(5.4.21) $$\|A_2\|_{L_p[\frac{s}{n_i+1},1-\frac{A}{n+1}]} \le M\ n^{-r}\|f^{(m)}\|_{p,2r} \qquad (1<p<\infty).$$

Next let $\frac{A}{n+1} \le x \le \frac{s}{n_i+1}$.

Since u varies between x and t, $U^{-r} \le X^{-r} + T^{-r}$.

Therefore,

$$A_2(x) \le (n_i+1)\ (1-x)^{n_i-m} \int_{\frac{A}{n+1}}^{\frac{s}{n_i+1}} |t-x|^{2r}(\ X^{-r} + T^{-r})$$

$$\frac{1}{|t-x|}\ \left|\int_x^t U^r |f^{(m+2r)}(u)|du\right|dt$$

Further, note that $\frac{A}{n+1} \le t, x \le \frac{s}{n_i+1}$ implies

$X^{-r} \le M\,(n+1)^r$,

$T^{-r} \le M\,(n+1)^r$ and

$|t-x|^{2r} \le M\,(n+1)^{-2r}$.

Thus,

$$A_2(x) \le M\ (n_i+1)\ (1-x)^{n_i-m}(n+1)^{-r}G(x) \int_{\frac{A}{n+1}}^{\frac{s}{n_i+1}} dt$$

$$\le M\ n^{-r}G(x).$$

As before $G(x)$ is the Hardy-Littlewood majorant of $X^r f^{(m+2r)}(x)$.

Therefore by lemma 1.6.2,

$$\|A_2\|_{L_p[\frac{A}{n+1},\ \frac{s}{n_i+1}]} \le M\ n^{-r}\|G\|_{L_p[\frac{A}{n+1},\ \frac{s}{n_i+1}]}$$

$$\le M\ n^{-r}\|G\|_p$$

$$\le M\ n^{-r}\|X^r f^{(m+2r)}\|_p \qquad (1<p<\infty)$$

(5.4.22) $$\le M\ n^{-r}\|f^{(m)}\|_{p,2r}.$$

The estimates (5.4.20-22) together amount to

$$\|S_o\|_{L_p[\frac{A}{n+1},1-\frac{A}{n+1}]} \le M\ n^{-r}\|f^{(m)}\|_{p,2r} \qquad (1<p<\infty).$$

Due to symmetry,

$$\|S_{n_i-m}\|_{L_p[\frac{A}{n+1},1-\frac{A}{n+1}]} \le M\, n^{-r}\|f^{(m)}\|_{p,2r} \qquad (1<p<\infty).$$

Thus establishing (5.4.17) and hence (5.4.18). Moreover in view of (5.4.4) the estimates (5.4.16&18) give

$$\|S_1\|_{L_p[\frac{A}{n+1},1-\frac{A}{n+1}]} \le M\, n^{-r}\|f^{(m)}\|_{p,2r} \qquad (1<p<\infty).$$

This completes case (i).

<u>case (ii)</u>: Let $\min(x,1-x) \le \frac{A}{n+1}$. Substituting

$$\sum_{l=1}^{r-1} \frac{1}{l!}(t-x)^l\, f^{(m+l)}(x) + \frac{1}{(r-1)!}\int_x^t (t-u)^{r-1} f^{(m+r)}(u)du.$$

for $f^{(m)}(t) - f^{(m)}(x)$ in the expression for $S_1(x)$,

$$S_1(x) = \frac{1}{m!}\sum_{s=0}^{m+1}\binom{m+1}{s}(-1)^{m+1-s}\sum_{l=1}^{r-1}\frac{1}{l!}\, f^{(m+l)}(x)\sum_{i=0}^{r-1} C(i,r)$$

$$\frac{(n_i+1)!}{(n_i-m)!}\sum_{k=0}^{n_i-m} p_{n_i-m,k}(x)\int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}}\left[\frac{k+s}{n_i+1} - t\right]^m (t-x)^l dt$$

$$+ \frac{1}{(r-1)!}\frac{1}{m!}\sum_{s=0}^{m+1}\binom{m+1}{s}(-1)^{m+1-s}\sum_{i=0}^{r-1} C(i,r)\frac{(n_i+1)!}{(n_i-m)!}.$$

$$\sum_{k=0}^{n_i-m} p_{n_i-m,k}(x)\int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}}\left[\frac{k+s}{n_i+1} - t\right]^m\int_x^t (t-u)^{r-1} f^{(m+r)}(u)dudt$$

(5.4.23) $= I_1(x) + I_2(x)$.

$I_1(x)$ can be shown, exactly in the same fashion as in the previous case, to be expressed as (5.4.6) except that now $1 \le l \le r-1$. Thus (5.4.7) and the remark 5.4.2 together imply

$$\|I_1\|_{L_p[\frac{A}{n+1},1-\frac{A}{n+1}]^c} \le M\, n^{-r} \|f^{(m)}\|_{p,2r} \qquad (1 \le p < \infty).$$

Remark 5.4.4: The above estimate holds for p = 1 as well.

In the following a similar estimate for $I_2(x)$ will be established.

$$|I_2(x)| \le M \sum_{s=0}^{m+1} \sum_{i=0}^{r-1} \frac{(n_i+1)!}{(n_i-m)!} \left(\frac{s}{n_i+1}\right)^m \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x)$$

$$\int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} \left|\int_x^t |t-u|^{r-1} |f^{(m+r)}(u)|du\right| dt$$

$$\le M \sum_{s=0}^{m+1} \sum_{i=0}^{r-1} (n_i+1) \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x)$$

$$\int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} \left|\int_x^t |t-u|^{r-1} |f^{(m+r)}(u)|du\right| dt$$

5.5 ORDER OF APPROXIMATION FOR $f \in W^m_{1,2r}$

In last section an order of approximation result for $f \in W^m_{p,2r}$, $1 < p < \infty$ was proved. Here we prove it for the case $p = 1$.

<u>THEOREM</u> <u>5.5.1</u>: Let $f \in W^m_{1,2r}$. Then

$$\|P^{(m)}_{n,r}f - f^{(m)}\|_1 \le M\, n^{-r}\, \|f^{(m)}\|_{1,2r}$$

<u>Proof</u>: Let $S(x) = P^{(m)}_{n,r}f(x) - f^{(m)}(x)$.

From (5.4.1),

(5.5.1) $\quad S(x) = S_1(x) + S_2(x)$

where

$$S_1(x) = \sum_{i=0}^{r-1} C(i,r)\, \frac{(n_i+1)!}{(n_i-m)!}\, \frac{1}{m!} \sum_{s=0}^{m+1} \binom{m+1}{s} (-1)^{m+1-s} \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x)$$
$$\int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} \left(\frac{k+s}{n_i+1} - t\right)^m \left(f^{(m)}(t) - f^{(m)}(x)\right)dt$$

and

$$S_2(x) = \sum_{i=0}^{r-1} C(i,r)\Bigg[\frac{(n_i+1)!}{(n_i-m)!}\, \frac{1}{m!} \sum_{s=0}^{m+1} \binom{m+1}{s} (-1)^{m+1-s} \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x)$$
$$\int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} \left(\frac{k+s}{n_i+1} - t\right)^m f^{(m)}(t)dt - 1 \Bigg] f^{(m)}(x).$$

By remark 5.4.1,

(5.5.2) $\quad \|S_2\|_1 \le M\, n^{-r} \|f^{(m)}\|_{1,2r}$.

As for $S_1(x)$ let

$$\int_0^1 |S_1(x)|dx = \int_0^{\frac{A}{n+1}} |S_1(x)|dx + \int_{\frac{A}{n+1}}^{1-\frac{A}{n+1}} |S_1(x)|dx + \int_{1-\frac{A}{n+1}}^{1} |S_1(x)|dx$$

$$= I_1 + I_2 + I_3 .$$

From (5.4.23),

$$S_1(x) = S_{11}(x) + S_{12}(x) .$$

where

$$S_{11}(x) = \frac{1}{m!} \sum_{s=0}^{m+1} \binom{m+1}{s} (-1)^{m+1-s} \sum_{l=1}^{2r-1} \frac{1}{l!} f^{(m+l)}(x) \sum_{i=0}^{r-1} C(i,r) .$$

$$\cdot \frac{(n_i+1)!}{(n_i-m)!} \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x) \int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} \left(\frac{k+s}{n_i+1} - t\right)^m (t-x)^l dt$$

and

$$S_{12}(x) = \frac{1}{(r-1)!} \frac{1}{m!} \sum_{s=0}^{m+1} \binom{m+1}{s} (-1)^{m+1-s} \sum_{i=0}^{r-1} C(i,r) \frac{(n_i+1)!}{(n_i-m)!} .$$

$$\cdot \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x) \int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} \left(\frac{k+s}{n_i+1} - t\right)^m \int_x^t (t-u)^{r-1} f^{(m+r)}(u)dudt$$

Now in view of remark 5.4.4,

$$\text{(5.5.3)} \qquad \int_0^{\frac{A}{n+1}} |S_{11}(x)|dx \leq M\, n^{-r} \|f^{(m)}\|_{1,2r} .$$

By (5.4.24),

$$|S_{12}(x)| = M \sum_{s=0}^{m+1} \sum_{i=0}^{r-1} (n_i+1) \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x) \int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} |t-x|^{r-1} \left|\int_x^t |f^{(m+r)}(u)|du\right| dt$$

$$\le M \, \|f^{(m+r)}\|_1 \sum_{s=0}^{m+1} \sum_{i=0}^{r-1} R(s,i,x)$$

Applying lemma 4.3.3 with n and m replaced by n_i+1 and $m+1$ respectively,

$$|R(s,i,x)| \le M \frac{1}{(n_i+1)^{r-1}} \left(\sum_{\nu=0}^{2r-2} \sum_{j=0}^{2r-2-\nu} |T_{n_i-m,j}(x)| \right)^{1/2}$$

$$(5.5.4) \qquad \le M \frac{1}{n^{r-1}} \left(\sum_{\nu=0}^{2r-2} \sum_{j=0}^{2r-2-\nu} |T_{n_i-m,j}(x)| \right)^{1/2}$$

Since $T_{n_i-m,0}(x) \equiv 1$ and $T_{n_i-m,1}(x) \equiv 0$ and for $j \ge 2$ by (4.1.3)

$$|T_{n_i-m,j}(x)| \le \sum_{k=1}^{[j/2]} |a_{k,j}(x)| \, ((n_i-m)X)^k$$

$$\le M \sum_{k=1}^{[j/2]} ((n+1)X)^k$$

$$\le M \qquad\qquad \text{(because } x \le A/(n+1)\text{)},$$

Hence,

$$|T_{n_i-m,j}(x)| \le M, \qquad 0 \le j \le 2(r-1)-\nu \text{ and } 0 \le \nu \le 2(r-1).$$

Thus,

$$(5.5.4) \qquad \int_0^{\frac{A}{n+1}} |S_{12}(x)|\,dx \le M\, n^{-r} \|f^{(m)}\|_{1,2r} .$$

The estimates (5.5.3-4) imply

$$I_1 \le M\, n^{-r} \|f^{(m)}\|_{1,2r} .$$

A similar analysis shows that

$$I_3 \le M\, n^{-r} \|f^{(m)}\|_{1,2r} .$$

As for I_2, from (5.4.4),

$$S_1(x) = S_{11}(x) + S_{12}(x)$$

where

$$S_{11}(x) = \frac{1}{m!} \sum_{s=0}^{m+1} \binom{m+1}{s} (-1)^{m+1-s} \sum_{l=1}^{2r-1} \frac{1}{l!} f^{(m+l)}(x) \sum_{i=0}^{r-1} C(i,r) \frac{(n_i+1)!}{(n_i-m)!}$$

$$\sum_{k=0}^{n_i-m} p_{n_i-m,k}(x) \int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} \left(\frac{k+s}{n_i+1} - t\right)^m (t-x)^l\,dt$$

and

$$S_{12}(x) = \frac{1}{(2r-1)!}\,\frac{1}{m!} \sum_{s=0}^{m+1} \binom{m+1}{s} (-1)^{m+1-s} \sum_{i=0}^{r-1} C(i,r)\, \frac{(n_i+1)!}{(n_i-m)!}$$

$$\cdot \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x) \int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} \left(\frac{k+s}{n_i+1} - t\right)^m \int_x^t (t-u)^{2r-1} f^{(m+2r)}(u)\,du\,dt$$

In view of the remark 5.4.3,

$$(5.5.5) \qquad \int_{\frac{A}{n+1}}^{1-\frac{A}{n+1}} |S_{11}(x)|\,dx \le M\, n^{-r} \|f^{(m)}\|_{1,2r} .$$

So what is required is an estimate for $S_{12}(x)$.

By (5.4.16),

$$|S_{12}(x)| \le M \sum_{s=0}^{m+1} \sum_{i=0}^{r-1} S(s,i,x)$$

where

$$S(s,i,x) = (n_i+1) \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x) \int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} \left|\int_x^t |t-u|^{2r-1} |f^{(m+2r)}(u)|du\right|dt$$

so in order to show that

$$\int_{\frac{A}{n+1}}^{1-\frac{A}{n+1}} |S_{12}(x)|dx \le M\, n^{-r} \|f^{(m)}\|_{1,2r} \tag{5.5.6}$$

it is sufficient to show that

$$\int_{\frac{A}{n+1}}^{1-\frac{A}{n+1}} |S(s,i,x)|dx \le M\, n^{-r} \|f^{(m)}\|_{1,2r} \tag{5.5.7}$$

$$(0 \le s \le m+1 \text{ and } 0 \le i \le r-1)$$

In $S(s,i,x)$ the terms corresponding to (i) $k = 0$ and (ii) $k=n_i-m$ when $s = m+1$ are treated separately . Let $S_o(s,i,x)$ be the term corresponding to $k = 0$,

$$S_o(s,i,x) = (n_i+1)\, p_{n_i-m,0}(x) \int_0^{\frac{s}{n_i+1}} \left|\int_x^t |t-u|^{2r-1} |f^{(m+2r)}(u)|du\right|dt$$

choosing $A = \frac{1}{C}$, where C is as in (1.5.1a),

$$\int_{\frac{A}{n+1}}^{1-\frac{A}{n+1}} |S_o(s,i,x)|dx = \left[\int_{\frac{A}{n+1}}^{\frac{s}{n_i+1}} + \int_{\frac{s}{n_i+1}}^{1-\frac{s}{n_i+1}} + \int_{\frac{s}{n_i+1}}^{1-\frac{A}{n+1}}\right] |S_o(s,i,x)|dx$$

$$= J_1 + J_2 + J_3$$

$$J_1 = \int_{\frac{A}{n+1}}^{\frac{s}{n_i+1}} (n_i+1)\,(1-x)^{n_i-m} \int_0^{\frac{A}{n+1}} \left|\int_x^t |t-u|^{2r-1} \frac{U^r|f^{(m+2r)}(u)|}{U^r}\,du\right| dtdx$$

$$+ \int_{\frac{A}{n+1}}^{\frac{s}{n_i+1}} (n_i+1)\,(1-x)^{n_i-m} \int_{\frac{A}{n+1}}^{\frac{s}{n_i+1}} \left|\int_x^t |t-u|^{2r-1} \frac{U^r|f^{(m+2r)}(u)|}{U^r}\,du\right| dtdx$$

$$= J_{11} + J_{12} .$$

For J_{11}, $0 \le t \le u \le x$ which implies $|t-u| \le u$ and $\frac{1}{1-u} \le \frac{1}{1-x}$.

Thus, using lemma 4.3.5,

$$J_{11} \le M \,\|U^r f^{(m+2r)}\|_1 \int_{\frac{A}{n+1}}^{\frac{s}{n_i+1}} (1-x)^{n_i-m} . x^{r-1} dx$$

$$\le M \frac{1}{(n_i-m)^{r-1}} \|U^r f^{(m+2r)}\|_1 \int_{\frac{A}{n+1}}^{\frac{s}{n_i+1}} dx$$

$$\le M\, n^{-r} \|f^{(m)}\|_{1,2r} .$$

As for J_{12}, $\frac{A}{n+1} \le x,t \le \frac{s}{n_i+1}$ implies

$$X^{-r} \le M\, n^r$$

$$T^{-r} \le M\, n^r \text{ and } \quad |t-x| \le M\, n^{-1}$$

Thus,

$$J_{12} \le M \,\|U^r f^{(m+2r)}\|_1 \int_{\frac{A}{n+1}}^{\frac{s}{n_i+1}} (n_i+1)\,(1-x)^{n_i-m}$$

$$\int_{\frac{A}{n+1}}^{\frac{s}{n_i+1}} |t-x|^{2r-1} (X^{-r} + T^{-r}) dtdx$$

$$\le M\, n^{-r+1}\, \|U^r f^{(m+2r)}\|_1 \int_{\frac{A}{n+1}}^{\frac{s}{n_i+1}} dx$$

$$\le M\, n^{-r} \|f^{(m)}\|_{1,2r}\,.$$

The estimates for J_{11} and J_{12} together amount to

$$J_1 \le M\, n^{-r} \|f^{(m)}\|_{1,2r}\,.$$

For J_3, $1-\frac{s}{n_i+1} \le x \le 1-\frac{A}{n+1}$ and $0 \le t \le \frac{s}{n_i+1}$ we have $0 \le t \le u \le x$.

Thus, $|t-u| \le u$ and $\frac{1}{1-u} \le \frac{1}{1-x} \le \frac{s}{n_i+1}$ and

$$J_3 = \int_{1-\frac{s}{n_i+1}}^{1-\frac{A}{n+1}} (n_i+1)(1-x)^{n_i-m} \int_0^{\frac{s}{n_i+1}} \left|\int_x^t |t-u|^{2r-1} \frac{U^r |f^{(m+2r)}(u)|}{U^r} du\right| dt\, dx$$

$$\le \int_{1-\frac{s}{n_i+1}}^{1-\frac{A}{n+1}} \|U^r f^{(m+2r)}\|_1 \; (n_i+1) \int_0^{\frac{s}{n_i+1}} x^{r-1} \frac{1}{(1-x)^r}\, dt\, dx$$

$$\le M\, n^{-r} \; \|f^{(m)}\|_{1,2r}$$

For J_2, $\frac{s}{n_i+1} \le x \le 1-\frac{s}{n_i+1}$ and $0 \le t \le \frac{s}{n_i+1}$ so again $0 \le t \le u \le x$.

Consequently, $|t-u| \le u$ and $\frac{1}{1-u} \le \frac{1}{1-x}$ and

$$J_2 = \int_{\frac{s}{n_i+1}}^{1-\frac{s}{n_i+1}} (n_i+1)(1-x)^{n_i-m} \int_0^{\frac{s}{n_i+1}} \left|\int_x^t |t-u|^{2r-1} \frac{U^r |f^{(m+2r)}(u)|}{U^r} du\right| dt\, dx$$

$$\le \int_{\frac{s}{n_i+1}}^{1-\frac{s}{n_i+1}} (n_i+1)\; (1-x)^{n_i-m-r} x^{r-1} \int_0^{\frac{s}{n_i+1}} \left|\int_x^t U^r |f^{(m+2r)}(u)| du\right| dt\, dx$$

$$\leq M \, \|U^r f^{(m+2r)}\|_1 \int_0^1 x^{r-1}(1-x)^{n_i-m-r} dx$$

$$\leq M \, n^{-r} \, \|f^{(m)}\|_{1,2r} \, .$$

The estimates for J_1, J_2 and J_3 together give

$$(5.5.10) \quad \int_{\frac{A}{n+1}}^{1-\frac{A}{n+1}} |S_o(s,i,x)|\,dx \leq M \, n^{-r} \, \|f^{(m)}\|_{1,2r} \, .$$

By symmetry

$$(5.5.11) \quad \int_{\frac{A}{n+1}}^{1-\frac{A}{n+1}} |S_{n_i-m}(m+1,i,x)|\,dx \leq M \, n^{-r} \, \|f^{(m)}\|_{1,2r} \, .$$

where $S_{n_i-m}(m+1,i,x)$ is the term corresponding to $k = n_i-m$ when $s = m+1$.

The remaining terns in $S(s,i,x)$ are

$$S_r(s,i,x) = (n_i+1)\sum_{k=1}^{n_i-m} p_{n_i-m,k}(x)\int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} \left|\int_x^t |t-u|^{2r-1} |f^{(m+2r)}(u)|\,du\right| dt$$

Let $\quad K(n_i,m,s,x,t) = (n_i+1) \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x) \, \chi_{k,s,n_i}(t) \, .$

where $\chi_{k,s,n_i}(t)$ is the characteristic function of the interval $[\frac{k}{n_i+1}, \frac{k+s}{n_i+1}]$. So,

$$S_r(s,i,x) = \int_{\frac{1}{n_i+1}}^{1-\frac{1}{n_i+1}} K(n_i,m,s,x,t) \left|\int_x^t |t-u|^{2r-1} |f^{(m+2r)}(u)|\,du\right| dt$$

$$\int_{\frac{A}{n+1}}^{1-\frac{A}{n+1}} S_r(s,i,x)dx \leq M\,(n+1)^{-r+\frac{1}{2}} . \sum_{j=0}^{p} \int_{\frac{A}{n+1}}^{1-\frac{A}{n+1}}$$

$$\int_{\frac{A}{n+1}}^{1-\frac{A}{n+1}} \chi_{x,j+1}(u)\; U^{r-\frac{1}{2}} \,|f^{(m+2r)}(u)|\,dudx$$

$$(5.5.12) \qquad = M\,(n+1)^{-r+\frac{1}{2}} . \sum_{j=0}^{p} \int_{\frac{A}{n+1}}^{1-\frac{A}{n+1}} U^{r-\frac{1}{2}} |f^{(m+2r)}(u)|\,du$$

$$\int_{\frac{A}{n+1}}^{1-\frac{A}{n+1}} \chi_{x,j+1}(u)dx$$

Fubini's theorem has been used to get the above.

Since $\chi_{x,j+1}(u)$ is the characteristic function of the interval

$[x-(j+1)\left(\frac{X}{n}\right)^{1/2}, x+(j+1)\left(\frac{X}{n}\right)^{1/2}]$, therefore the variation of u is

$$(5.5.13) \quad x - (j+1)\left(\frac{X}{n}\right)^{1/2} \leq u \leq x + (j+1)\left(\frac{X}{n}\right)^{1/2}$$

An upper bound for the range of x for fixed n and u will be found in the following .

From (5.5.13),

$$(x-u)^2 = \theta.(j+1)^2 \frac{X}{n}, \qquad 0 \leq \theta \leq 1$$

equivalently,

$$x^2(1 + \frac{\theta}{n}(j+1)^2) - x(2u + \frac{\theta}{n}(j+1)^2) + u^2 = 0 .$$

This shows that for a fixed value of n and u, two values of x are determined . Consequently the range of x is contained between the maximum and minimum roots of the above quadratic equation as θ varies between 0 and 1. The roots of the quadratic are

$$\frac{2u + \frac{\theta}{n}(j+1)^2 \mp \sqrt{\left(2u + \frac{\theta}{n}(j+1)^2\right)^2 - 4u^2\left(1 + \frac{\theta}{n}(j+1)^2\right)}}{2\left(1 + \frac{\theta}{n}(j+1)^2\right)}$$

which can be shown to be equal to

$$\frac{2u + \frac{\theta}{n}(j+1)^2 \mp \sqrt{\left(\frac{\theta}{n}\right)^2 (j+1)^4 + 4\,\frac{\theta}{n}(j+1)^2 U}}{2\left(1 + \frac{\theta}{n}(j+1)^2\right)}$$

let, $$\alpha = \sup_{0 \le \theta \le 1} \frac{2u + \frac{\theta}{n}(j+1)^2 + \sqrt{\left(\frac{\theta}{n}\right)^2 (j+1)^4 + 4\,\frac{\theta}{n}(j+1)^2 U}}{2\left(1 + \frac{\theta}{n}(j+1)^2\right)}$$

and $$\beta = \inf_{0 \le \theta \le 1} \frac{2u + \frac{\theta}{n}(j+1)^2 - \sqrt{\left(\frac{\theta}{n}\right)^2 (j+1)^4 + 4\,\frac{\theta}{n}(j+1)^2 U}}{2\left(1 + \frac{\theta}{n}(j+1)^2\right)}$$

So,

$$\alpha \le u + \frac{(j+1)^2}{n} + (j+1)\left(\frac{U}{n}\right)^{1/2}$$

and $$\beta \ge \frac{u}{1 + \frac{(j+1)^2}{n}} - \frac{\sqrt{\frac{(j+1)^4}{n^2} + 4\,\frac{U}{n}(j+1)^2}}{2}$$

Therefore the measure of the range of $x \le \alpha - \beta$

$$\le \frac{3}{2}\,\frac{(j+1)^2}{n} + u\,\frac{(j+1)^2}{n+(j+1)^2} + (j+1)\left(\frac{U}{n}\right)^{1/2}$$

$$\le M\,\frac{1}{n+1} + \left(\frac{U}{n}\right)^{1/2}$$

$$\le \left(\frac{U}{n}\right)^{1/2}\left[\,M\left(\frac{1}{(n+1)U}\right)^{1/2} + 1\right]$$

$$\le M\left(\frac{U}{n}\right)^{1/2} \qquad \left(\text{because } \frac{A}{n+1}<u<1-\frac{A}{n+1}\right)$$

Therefore from (5.5.12),

$$\int_{\frac{A}{n+1}}^{1-\frac{A}{n+1}} |S_r(s,i,x)|\,dx \le M\,n^{-r}\sum_{j=0}^{p}\int_{\frac{A}{n+1}}^{1-\frac{A}{n+1}} U^r\,|f^{(m+2r)}(u)|\,du$$

$$\le M\,n^{-r}\,\|f^{(m)}\|_{1,2r}\,.$$

This together with (5.5.10-11) establishes (5.5.9) completely and hence (5.5.8) . Thus in view of (5.5.7-8),

$$\int_{\frac{A}{n+1}}^{1-\frac{A}{n+1}} |S_1(x)|\,dx \le M\,n^{-r}\,\|f^{(m)}\|_{1,2r}\,.$$

Hence the theorem . ■

<u>Corollary 5.5.1</u>: Let $r,\ m \in \mathbb{N}$ and $f \in W^m_{1,2r}$, then

$$\|P_{n,r}f-f\|_{m,1} \le M\,n^{-r}\|f\|_{m,1,2r}\,.$$

Proof: Winslin[92] showed that for $r \in \mathbb{N}$ and $f \in L_{1,2r}$

$$\|P_{n,r}f-f\|_1 \le M\, n^{-r}\|f\|_{1,2r} \,.$$

Using lemma 5.1.5 and theorem 5.5.1 together with this we get the corollary . ∎

5.6 ORDER OF APPROXIMATION , THE p = ∞ CASE

Following Winslin[92] we have

$$L_{\infty,2r} = \left\{ f \in L_\infty[0,1] : \begin{array}{l} f^{(2r-1)} \in \text{loc. A.C.}(0,1) \text{ and} \\ f^{(r)}\ ,\ X^r f^{(2r)} \in L_\infty[0,1] \end{array} \right\} .$$

Let,

$$W^m_{\infty,2r} = \left\{ f \in W^m_\infty[0,1] : f^{(m)} \in L_{\infty,2r} \right\}$$

and

$$\|f\|_{m,\infty,2r} = \|f\|_{m,\infty} + \|f^{(m+r)}\|_\infty + \|X^r f^{(m+2r)}\|_\infty$$

where $\|f\|_\infty = \operatorname*{ess.\,sup}_{x\in[0,1]} |f(x)|$ and $\|f\|_{m,\infty} = \|f\|_\infty + \|f^{(m)}\|_\infty$.

That the spaces $W^m_{\infty,2r}$ are complete with respect to the norm $\|\cdot\|_{m,\infty,2r}$ follows from theorem 4.1.1 .

The space $W^{m,\phi}_{\infty,2r}$, $\phi \in \Phi_{2r}$ is defined as follows

$$W^{m,\phi}_{\infty,2r} = \left\{ f \in W^m_\infty : K_m(t^{2r},f) = O(\phi(t)) \right\}$$

where

$$K_m(t,f) = \inf_{g\in W^m_{\infty,2r}} \left\{ \|f-g\|_{m,\infty} + t\|g\|_{m,\infty,2r} \right\} .$$

<u>LEMMA</u> <u>5.6.1</u>: Let $m,r \in \mathbb{N}_o$, then

$$W^m_{\infty,2r} \subseteq W^o_{\infty,2r} \equiv L_{\infty,2r}$$

moreover, $\|f\|_{\infty,2r} \le M\ \|f\|_{m,\infty,2r}$.

where $\|f\|_{\infty,2r} = \|f\|_\infty + \|f^{(r)}\|_\infty + \|X^r f^{(2r)}\|_\infty$.

<u>Proof</u>: For m=r=0 the lemma is obviously true . So let $m,r \ge 1$. Now $f \in W^m_{\infty,2r}$ implies that $f^{(m+2r-1)} \in$ loc A.C.(0,1), $f^{(m+r)} \in L_\infty[0,1]$ and $X^r f^{(m+2r)} \in L_\infty[0,1]$. In order to show that $f \in L_{\infty,2r}$ we must have that $f^{(2r-1)} \in$ loc A.C.(0,1), which of course is true, and $f^{(r)}, X^r f^{(2r)} \in L_\infty[0,1]$. Now, since f and $f^{(m+r)} \in L_\infty[0,1]$, applying lemma 1.6.1,

$$\|f^{(r)}\|_\infty \le M(\|f\|_\infty + \|f^{(m+r)}\|_\infty) .$$

Thus, $f^{(r)} \in L_\infty[0,1]$. To show that $X^r f^{(2r)} \in L_\infty[0,1]$ consider two cases (i) $m \ge r$ and (ii) $1 \le m < r$. Let $m \ge r$, then $m+r \ge 2r$ and therefore by lemma 1.6.1,

$$\|f^{(2r)}\|_\infty \le M(\|f\|_\infty + \|f^{(m+r)}\|_\infty)$$

moreover, since $X \le \frac{1}{4}$,

$$\|X^r f^{(2r)}\|_\infty \le M\ \|f^{(2r)}\|_\infty \le M(\|f\|_\infty + \|f^{(m+r)}\|_\infty) .$$

Thus,

$$\|f\|_{\infty,2r} \le M\ \|f\|_{m,\infty,2r} .$$

Next, let $1 \le m < r$, in this case applying lemma 4.1.4,

$$\|X^r f^{(2r)}\|_\infty \le \left(\frac{1}{4}\right)^m \|X^{r-m} f^{(m+2r-m)}\|_\infty \le M(\|f^{(m)}\|_\infty + \|f^{(m+2r)}\|_\infty) .$$

Thus,

$$\|f\|_{\infty,2r} \le M\ \|f\|_{m,\infty,2r} .$$

Hence the lemma . ∎

Analogues of Bernstein type inequalities and that of the order of approximation for $f \in W^m_{\infty,2r}$ will be proved in the following.

THEOREM 5.6.1: Let $r \in \mathbb{N}$. Then

$$(5.6.1) \qquad \|P_n^{(m)} f\|_{\infty,2r} \le M\, n^r\, \|f^{(m)}\|_\infty, \quad f \in W^m_\infty$$

$$(5.6.2) \qquad \|P_n^{(m)} f\|_{\infty,2r} \le M\, \|f^{(m)}\|_{\infty,2r}, \quad f \in W^m_{\infty,2r}$$

and

$$(5.6.3) \qquad \|P_{n,r}^{(m)} f - f^{(m)}\|_\infty \le M\, n^{-r}\, \|f^{(m)}\|_{\infty,2r}, \quad f \in W^m_{\infty,2r}\,.$$

Proof: In view of the fact that

$$P_n^{(m)} f(x) = B_{n+1}^{(n+1)} F(x)$$

where $F(x) = \int_0^x f(t)dt$, the assertions (5-6.1-2) follow immediately from the corollaries 4.2.1-2. As for (5.6.3), it will also follow quite easily from what has been proved in the previous chapter and the earlier sections of this chapter.

Let $S(x) = P_{n,r}^{(m)} f(x) - f^{(m)}(x)$.

Following (5.4.1),

$$S(x) = S_1(x) + S_2(x)$$

where

$$S_1(x) = \sum_{i=0}^{r-1} C(i,r) \frac{(n_i+1)!}{(n_i-m)!} \frac{1}{m!} \sum_{s=0}^{m+1} \binom{m+1}{s} (-1)^{m+1-s} \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x)$$

$$\int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} \left(\frac{k+s}{n_i+1} - t \right)^m (f^{(m)}(t) - f^{(m)}(x))dt$$

and

$$S_2(x) = \sum_{i=0}^{r-1} C(i,r)\left[\frac{(n_i+1)!}{(n_i-m)!} \frac{1}{m!} \sum_{s=0}^{m+1} \binom{m+1}{s} (-1)^{m+1-s} \sum_{k=0}^{n_i-m} p_{n_i-m,k}(x) \int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} \left(\frac{k+s}{n_i+1} - t\right)^m f^{(m)}(t)dt - 1 \right] f^{(m)}(x).$$

From (5.4.3),

$$\|S_2\|_{L_\infty[0,1]} \le M\, n^{-r} \|f^{(m)}\|_{L_\infty[0,1]}$$

$$\le M\, n^{-r} \|f^{(m)}\|_{\infty,2r} \tag{5.6.4}$$

The term $S_1(x)$ will be analyzed in two parts

<u>Case</u> <u>i</u>: Let $\frac{A}{n+1} \le x \le 1 - \frac{A}{n+1}$

As in case (i) of theorem 5.4.1

$$S_1(x) = S_{11}(x) + S_{12}(x),$$ (by (5.4.4))

where $S_{11}(x)$ and $S_{12}(x)$ are as in (5.4.4).

In the case $1 \le p < \infty$, in order to establish that

$$\|S_{11}\|_{L_p[\frac{A}{n+1},\,1-\frac{A}{n+1}]} \le M\, n^{-r} \|f^{(m)}\|_{p,2r}$$

The properties of the spaces $W^m_{p,2r}$, $1 \le p < \infty$ that were made use of were

$$\|X^{r-i} f^{(m+2r-i)}\|_{L_p[0,1]} \le \|f^{(m)}\|_{p,2r}, \quad i = 0,1,\ldots, r.$$

and

$$\|f^{(m+i)}\|_{L_p[0,1]} \le \|f^{(m)}\|_{p,2r}, \quad i = 0,1,\ldots, r.$$

But, in view of lemmas 4.1.4-5, these hold good even for the case $p = \infty$ in the way $W^m_{\infty,2r}$ and the $\|\ \|_{\infty,2r}$ have been defined.

Hence following exactly along the lines of theorem 5.4.1.

$$\|S_{11}\|_{L_\infty[\frac{A}{n+1},1-\frac{A}{n+1}]} \le M\, n^{-r}\, \|f^{(m)}\|_{\infty,2r}$$

As for $S_{12}(x)$, observe that it differs from the remainder term $S_{12}(x)$ of theorem 4.3.1 Case (i) only in that n, n_1 and m appearing there have been replaced by $n+1$, n_1+1 and $m+1$ respectively. Hence a similar analysis allows

$$\|S_{12}\|_{L_\infty[\frac{A}{n+1},1-\frac{A}{n+1}]} \le M\, n^{-r}\, \|f^{(m)}\|_{\infty,2r}$$

The estimates for $S_{11}(x)$ and $S_{12}(x)$ together give

$$(5.6.5) \qquad \|S_1\|_{L_\infty[\frac{A}{n+1},1-\frac{A}{n+1}]} \le M\, n^{-r}\, \|f^{(m)}\|_{\infty,2r} .$$

<u>case ii</u>: $\min(x,1-x) \le \frac{A}{n+1}$. From (5.4.23),

$$S_1(x) = \bar{S}_{11}(x) + \bar{S}_{12}(x)$$

where $\bar{S}_{11}(x)$ and $\bar{S}_{12}(x)$ are same as $I_1(x)$ & $I_2(x)$ in (5.4.23) respectively.

Again for the same reasons, as put forward for the case(i), theorem 5.4.1(case ii) gives that

$$\|\bar{S}_{11}\|_{L_\infty[\frac{A}{n+1},1-\frac{A}{n+1}]^c} \le M\, n^{-r}\, \|f^{(m)}\|_{\infty,2r}$$

and theorem 4.3.1 case (ii) gives

and $\|\bar{S}_{12}\|_{L_\infty[\frac{A}{n+1},1-\frac{A}{n+1}]^c} \le M\, n^{-r}\, \|f^{(m)}\|_{\infty,2r}$.

Thus,

$$(5.6.6) \quad \|S_1\|_{L_\infty[\frac{A}{n+1},1-\frac{A}{n+1}]^c} \le M\, n^{-r}\, \|f^{(m)}\|_{\infty,2r} .$$

Now (5.6.3) follows from (5.6.4-6) .

Hence the theorem .

$$\le \left\{ C_3\ \phi(n^{-1/2}) + t^{2r}\left\{ \|P_{n,r}^{(m)}(f-g)\|_{p,2r} + \|P_{n,r}^{(m)}g\|_{p,2r} \right\} \right\}$$

where g is such that $g^{(m)} \in L_{p,2r}$.

Since $\|P_{n,r}^{(m)}f\|_{p,2r} \le M \max\limits_{0\le i\le r-1} \|P_{n_i}^{(m)}f\|_{p,2r}$, corollaries 5.2.1 and 5.3.1 along with (5.6.1-2) and (1.5.1a), give

$$\|P_{n,r}^{(m)}(f-g)\|_{p,2r} \le M\ n^r\ \|f^{(m)}-g^{(m)}\|_p \qquad (1\le p\le\infty)$$

$$\|P_{n,r}^{(m)}g\|_{p,2r} \qquad \le M\ \|g^{(m)}\|_{p,2r} \qquad (1\le p\le\infty)$$

Therefore,

$$K(t^{2r},f^{(m)}) \le C_4 \left\{ \phi(n^{-1/2}) + t^{2r}\left\{ n^r\|f^{(m)}-g^{(m)}\|_p + \|g^{(m)}\|_{p,2r}\right\}\right\}$$

$$\le C_5 \left\{ \phi(n^{-1/2}) + t^{2r}n^rK(n^{-r},f^{(m)}) \right\}$$

Now (iv) follows from lemma 1.6.3.

iv) implies i)

Suppose $K(t^{2r},f^{(m)}) = O(\phi(t))$, therefore for given t there is a h_t in $L_{p,2r}$ such that

$$(5.7.5) \qquad \|f^{(m)}-h_t\|_p \le C_6\ \phi(t)$$

$$(5.7.6) \qquad t^{2r}\|h_t\|_{p,2r} \le C_6\ \phi(t)\ .$$

Let

$$g_t(x) = \int_0^x \int_0^{s_1} \cdots \int_0^{s_{m-1}} h_t(s)ds\,ds_{m-1}\cdots ds_1$$

$$+ f(0) + xf'(0) + \ldots + \frac{x^{m-1}}{(m-1)!}\ f^{(m-1)}(0)$$

Moreover,

$$f(x) = \int_0^x \int_0^{s_1} \ldots \int_0^{s_{m-1}} f^{(m)}(s)ds ds_{m-1} \ldots ds_1$$

$$+ f(0) + xf'(0) + \ldots + \frac{x^{m-1}}{(m-1)!} f^{(m-1)}(0)$$

Clearly,

$$\|f^{(m)} - g_t^{(m)}\|_p = \|f^{(m)} - h_t\|_p$$

Also,

$$|f(x) - g_t(x)| \le \int_0^1 |f^{(m)}(s) - h_t(s)|ds$$

$$\le \|f^{(m)} - h_t\|_p$$

Therefore by (5.7.5),

(5.7.7) $\quad \|f - g_t\|_{m,p} \le C_6 \, \phi(t)$.

Further,

$$|g_t(x)| \le \int_0^1 |h_t(s)|ds + \sum_{i=0}^{m-1} \frac{1}{i!} |f^{(i)}(0)|$$

$$\le \|h_t\|_p + C_7$$

Consequently,

$$\|g_t\|_p \le \|h_t\|_p + C_7$$

Now using (4.4.3) and (5.7.6) it could be seen that

$$\frac{t^{2r}}{\phi(t)} \|g_t\|_p \le \frac{t^{2r}}{\phi(t)} \|h_t\|_p + C_7 \frac{t^{2r}}{\phi(t)}$$

$$\le C_8$$

This alongwith (5.7.6) gives

(5.7.8) $$t^{2r}\|g_t\|_{m,p,2r} = t^{2r}\|g_t\|_p + t^{2r}\|g_t^{(m)}\|_{p,2r} \le C_9\ \phi(t)$$

Inequalities (5.7.7-8) imply

$$K_m(t^{2r},f) = O(\phi(t))$$

This completes the proof of (iv) implies (i) .

Hence the theorem . ∎

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Thus, in view of (1.5.1a,b),

$$(5.4.12)\quad |S(n,m,l,\mu,j;f,x)| \le M\, n^{-r} \sum_{k=1}^{[j/2]} X^k |f^{(m+l)}(x)|.$$

Moreover since $\sum_{k=1}^{[j/2]} X^k$ is bounded for all x, lemma 5.1.4 gives

$$(5.4.13)\quad \|S(n,m,l,\mu,j;f,x)\|_p \le M\, n^{-r}\, \|f^{(m)}\|_{p,2r}\quad ,\ (2 \le l \le r)$$

Next let $r+1 \le l \le 2r-1$ again there are two possibilities (i) $l+\mu-q < r$ and (ii) $l+\mu-q \ge r$. Suppose $l+\mu-q < r$.

By (5.4.12),

$$|S(n,m,l,\mu,j;f,x)| \le M\, n^{-r} X^{r-\beta} |f^{(m+2r-\beta)}(x)| \sum_{k=1}^{[j/2]} X^{k+\beta-r}$$

where $\beta = r+\mu+\alpha-q$ and $\alpha \in \mathbb{N}$ is such that $l+\mu-q = r-\alpha$.

Now the choice of l implies $1 \le \beta \le r-1$ also $\sum_{k=1}^{[j/2]} X^{k+\beta-r}$ is bounded for all $x \in [0,1]$ this is because $k+\beta-r \ge 0$. Therefore, by lemma 5.1.3,

$$(5.4.14)\quad \|S(n,m,l,\mu,j;f,x)\|_p \le M\, n^{-r}\, \|f^{(m)}\|_{p,2r},\quad 1 \le p < \infty.$$

Next suppose $l+\mu-q \ge r$. Here using (5.4.11),

$$|S(n,m,l,\mu,j;f,x)| = |f^{(m+l)}(x)| \sum_{i=0}^{r-1} |C(i,r)| \sum_{k=1}^{[j/2]} |a_{k,j}(x)|\, X^k \sum_{q=0}^{k} \binom{k}{q} (m+1)^{k-q}\, (n_i+1)^{-l-\mu+q}$$

$$\le M \sum_{k=1}^{[j/2]} \sum_{q=0}^{k} (n+1)^{-l-\mu+q}\, X^k |f^{(m+l)}(x)|$$

The last inequality follows due to the fact that $a_{l,j}(x)$ is bounded, $x \in [0,1]$, independent of n_i-m and then using (1.5.1a). But the choice of x implies that $nX \le A$. Therefore,

$$|T_{n_i-m,j}(x)| \le M , \quad 0\le j\le 2r-\nu \text{ and } 0\le\nu\le 2r.$$

Thus, establishing (5.4.25) now (5.4.24) and lemma 1.6.2 give

$$\begin{aligned}
\|I_2\|_{L_p[\frac{A}{n+1},1-\frac{A}{n+1}]^c} &\le M\, n^{-r}\, \|G\|_{L_p[\frac{A}{n+1},1-\frac{A}{n+1}]^c} \\
&\le M\, n^{-r}\, \|G\|_p . \\
&\le M\, n^{-r} \|f^{(m+r)}\|_p , \qquad 1 < p < \infty \\
&\le M\, n^{-r} \|f^{(m)}\|_{p,2r} .
\end{aligned}$$

The estimates for $I_1(x)$ and $I_2(x)$ together complete case(ii). Hence the theorem.■

<u>Corollary 5.4.1</u>: Let r, m ∈ ℕ and $f \in W^m_{p,2r}$, $1 < p \infty$, then

$$\|P_{n,r}f-f\|_{m,p} \le M\, n^{-r} \|f\|_{m,p,2r} .$$

<u>Proof</u>: Winslin [92] proved that for r ∈ ℕ and $f \in L_{p,2r}$

$$\|P_{n,r}f-f\|_p \le M\, n^{-r} \|f\|_{p,2r}, \quad (1<p<\infty).$$

This along with lemma 5.1.5 and theorem 5.4.1 gives the corollary. ■

where $\chi_{x,\,j+1}(u)$ is the characteristic function of the interval

$[x-\ (j+1)\left(\frac{X}{n}\right)^{1/2},\ x+\ (j+1)\left(\frac{X}{n}\right)^{1/2}]$.

Thus,

$$S_r(s,i,x) \le \int_{\frac{1}{n_i+1}}^{1-\frac{1}{n_i+1}} K(n_i,m,s,x,t)|t-x|^{2r-1}\ (\ X^{r-\frac{1}{2}} + T^{r-\frac{1}{2}}\ )dt$$

$$\sum_{j=0}^{p} \int_{\frac{A}{n+1}}^{1-\frac{A}{n+1}} \chi_{x,\,j+1}(u)\ U^{r-\frac{1}{2}}\ |f^{(m+2r)}(u)|du\ .$$

Let,

$$H_{n_i}(x) = \int_{\frac{1}{n_i+1}}^{1-\frac{1}{n_i+1}} K(n_i,m,s,x,t)|t-x|^{2r-1}\ (\ X^{r-\frac{1}{2}} + T^{r-\frac{1}{2}}\ )dt$$

$$= (n_i+1)\sum_{k=1}^{n_i-m} p_{n_i-m,k}(x)\int_{\frac{k}{n_i+1}}^{\frac{k+s}{n_i+1}} |t-x|^{2r-1}\ (\ X^{r-\frac{1}{2}} + T^{r-\frac{1}{2}}\ )dt$$

Recall that $k < n_i-m$ when $s = m+1$ also (1.5.1a) implies $\frac{A}{n_i+1} \le \frac{A}{n+1} < x < 1 - \frac{A}{n+1} \le 1-\frac{A}{n_i+1}$, therefore, using lemma 4.3.4, with $\alpha = 2r-1$, $\beta = r-\frac{1}{2}$, and n and m replaced by n_i+1 and $m+1$ respectively, we get

$$|H_{n_i}(x)| \le M\ (n_i+1)^{-r+\frac{1}{2}} \le M\ (n+1)^{-r+\frac{1}{2}}\ .$$

Consequently,

Moreover by lemma 5.1.2,

(5.7.2) $\|P_{n,r}f\|_{m,p} \le M \max_{0\le r-1} \|P_{n_i} f\|_p \le M \|f\|_{m,p}$, $f \in W_p^m[0,1]$

Now $f \in W_{p,2r}^{m,\phi}$ implies that

$K_m(t^{2r},f) \le C_1 \phi(t)$,

therefore for a given t there exists a $g_t \in W_{p,2r}^m$ such that

(5.7.3) $\|f-g_t\|_{m,p} \le C_1 \phi(t)$ and

(5.7.4) $\|g_t\|_{m,p,2r} \le C_1 t^{-2r}\phi(t)$.

Now,

$$\|P_{n,r}f-f\|_{m,p} \le \|P_{n,r}(f-g_t)\|_{m,p} + \|P_{n,r}g_t-g_t\|_{m,p} + \|g_t-f\|_{m,p}$$

Thus, by (5.7.1-4),

$$\|P_{n,r}f-f\|_{m,p} \le (M+1)\|g_t-f\|_{m,p} + M n^{-r}\|g_t\|_{m,p,2r}$$

$$\le C_2 (\phi(t) + n^{-r}t^{-2r}\phi(t))$$

taking $t = n^{-1/2}$ (ii) follows .

(ii) implies (iii) is obvious .

(iii) implies (iv)

Let $\|P_{n,r}^{(m)}f-f^{(m)}\|_p = O(\phi(n^{-1/2}))$, we have

$$K(t^{2r},f^{(m)}) = \inf_{g\in L_{p,2r}} \left\{ \|f^{(m)}- g\|_p + t \|g\|_{p,2r} \right\}$$

$$\le \left\{ \|P_{n,r}^{(m)}f-f^{(m)}\|_p + t^{2r}\|P_{n,r}^{(m)}f\|_{p,2r}\right\}$$